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COMPOSITES DESIGN

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This report includes the fundamental equations for the stiffness and strength of laminated composites and two approaches to determine the optimum strength of laminates. Formulas for micromechanics and for unsymmetric laminates are simplified so that material and geometric factors can be more easily optimized than the unabridge theory. This report is intended as a source of information for designers, fabricators and researchers of composites. Originator supplied keywords include:

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FOREWORD

This report was prepared under a joint sponsorship of the Mechanics and Surface Interactions Branch (AFWAL/MLBM), Nonmetallic Materials Division, Materials Laboratory, Air Force Wright Aeronautical Laboratories, Wright-Patterson Air Force Base, Ohio, and the Commissariat à l'Energie Atomique, 91680 Bruyères Le Chatel, France. The work of S. W. Tsai was performed under the support of Project Number 2307, "Nonmetallic Structural Materials", Task Number 2307P2, "Composite Materials and Mechanics Technology."

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Section 1

INTRODUCTION

1.1 BACKGROUND – The rapidly expanding applications of composites in the recent past have provided much optimism for the future of our technology. Although man-made composites have existed for thousands of years, the high technology of composites has evolved in the aerospace industry only in the last twenty years. Filament-wound pressure vessels using glass fibers were the first strength critical application for modern composites.

Then came boron filaments in the 1960's which started many US Air Force programs to promote aircraft structures made of composites. The F-111 horizontal stabilizer was the first flight-worthy composite component.

Production of composite stabilizer for the F-14 was another major milestone. That was followed by the composite stabilator for the F-15, and composite rudder and stabilizer for the F-16. More recently, Boeing 767 has nearly two tons of composite materials in its floor beams and all of its control surfaces.

Materials and processing advances have been instrumental to the growth of our technology. Graphite and aramid fibers became commercially available in the early 1970's. Epoxies are available for various use conditions. More recently, higher-temperature matrix materials and thermoplastics have emerged for more demanding applications of the future.

In the mean time, the high technology of composites has spurred applications outside the aerospace industry. The sporting goods is a major outlet of our material. Hundreds of tons of graphite composites were used for tennis and squash rackets, and golf shafts in 1983. These rackets and composites are synonymous. Such deep penetration of composites in a product is good for our technology. We should dedicate ourselves to promote composites for as many products as possible.

It is our belief that the acceptance of composites can be greatly enhanced if the cost is lowered and the design, simplified. We would like to address the design issue directly, and the cost indirectly.

1.2 DESIGNING WITH COMPOSITES

– Designing with any material has been more of an art than science; that with composites is no exception and has much to learn. Universities prefer teaching analysis to design. Books on analysis outnumber those on design by a wide margin. Research topics are rarely design-oriented. But products are made with or without rational design. Netting analysis is still considered useful for design. Same holds true for the carpet plot. Furthermore, design limit is based on some strain level; one level for laminates without holes, and a reduced level for those with holes. These are typical practices in industry. They are not rationally developed. In fact many of them are wrong. They are still used not because the theory is right but limited empirical data are available. Above all modern composites are so strong that no major failures have occurred.

Our desire is to use as much science as possible for designing with composites. For this reason, netting analysis, carpet plots, and limit strains will not be used. We rely on the laminated plate theory. For failure criteria, we will use the interacting quadratic criteria rather than the noninteracting stress or strain criteria. But the validity of our approach in design does not depend on the failure criterion chosen.

To us rationality is as important as practicality. We must have both if we are to succeed. We cannot afford to penalize composites by using the wrong design, or to deny ourselves of the extraordinary properties of composite materials.

As we see it, the basic issue in designing with composites is to learn to use the directionally dependent properties. The scalar approach for the design of isotropic materials is acceptable because stiffness and strength can each be represented by one parameter; i.e., the Young's modulus and the uniaxial strength, respectively. Poisson's ratio can be assumed to remain at 0.3. Strength under combined stresses such as the von Mises and Tresca criteria needs only one strength data, the uniaxial or shear strength, or any strength under combined stresses.

But for composites, the number of constants increase to four for the stiffness and at least five for the strength of an on-axis unidirectional ply. When we build a multidirectional laminate, the stiffness constants can be as many as 21, and the strength is five times the number of ply groups. We must use matrix in place of scalar operations. This is the challenge that all of us must face when we work with anisotropic materials.

How can we make our design conceptually simple and analytically proper? We can do it through modern computers where data, calculations and graphics can be fully integrated. In addition user-friendly programs require little or no computer experience. But we must also realize that the back-of-the-envelope calculations is still needed. We must know how to do the calculations by hand and by machine. This is our approach to designing with composites.

1.3 OUR APPROACH – The most outstanding feature of composites is the high specific stiffness and strength. The objective of design is to capitalize this feature. The most efficient configuration is the unidirectional composite. We will develop the method of the use of on- and off-axis unidirectional composite to carry combined loads.

If the loads are such that unidirectional composites are inadequate and inefficient, we will go to bi-directional laminates. The process continues as we increase the ply angles to 3, 4 and higher. Obviously, the number of angles must be balanced between considerations of manufacturing and cost, and the requirement for stiffness and strength. We take the highly directional composites as the upper bound; the quasi-isotropic laminate, the lower bound.

Our design approach can be divided in two categories:

- The principal stress approach for the single load case.
- The ranking approach for the multiple load case.

Although these are not all inclusive, they can be easily mastered and bring out the salient features of composite materials. The two approaches are in face complimentary and can be used sequentially or iteratively.

Matrix inversions are involved in the determination of laminate stiffness. It impossible to anticipate the effects of adding and subtracting plies or of rotating the entire laminate. These effects can be systematically established and should not be surprises. Instead of guessing or using intuition we recommend calculation.

In order to enhance confidence in our design calculations, we make constant comparison of our optimum composite with the quasi-isotropic laminate of the same composite as a lower bound, and with aluminum. We also make sure that calculations can be easily and accurately performed. *We see the personal computer as the most effective tool to aid our design.* In addition, formulas must be simplified, and the number of design variables reduced. The use of micromechanics, for example, reduces the number of material and geometric variables. The simplified theory of unsymmetric laminates makes design of such laminates simple without the inversion of 6x6 matrices. Finally notation can also be intimidating. We would like to present to you our system in the next sub-section.

1.4 OUR NOTATION – We try to follow contracted notation consistently like that in our textbook: *Introduction to Composite Materials*, by S. W. Tsai and H. T. Hahn, Technomic, Lancaster, Pennsylvania (1980). We are bound by the following rules:

- Single subscript for second-rank tensors; double subscripts for fourth-rank tensors. Shear component for stress and strain has only single subscript.

- Engineering shear strain is used. Same applies to the twisting curvature-displacement relation; i.e.,

  $$k_s = -2\partial^2 w/\partial x\partial y \qquad (1.1)$$

  Note that the factor of 2 for the relation is required here for the equivalent of the engineering shear strain. We need the negative sign for this and the two normal curvature relations.

- Letter subscripts x, y, z, t, u and s designate on-axis coordinates; numeric subscripts 1, 2, 3, 4, 5 and 6 designate off-axis coordinates.

- Engineering constants are defined from the components of the normalized compliance. Definitions are also given for the unsymmetric laminates. The coupling coefficients are normalized by columns, not by rows; Poisson's ratios are defined differently from other authors.

- Other symbols include:

  Asterisk [*] means normalized variable.
  Prime ['] means compressive or negative.
  Super o [^{o}] means in-plane.
  Super f [^{f}] means flexural.

- We try to use normalized variables (those with *) for properties in addition to the absolute. Normalized properties are easy to compare with one another. Stiffness components for example are expressed in Pa.

- We try to use dimensionless variables whenever possible. Then we do not have to worry about SI versus English units. We would use number of plies to represent thickness in stead of length which must of a unit.

1.5 COMPUTERS - Modern computers play a major role in designing with composites. Personal computers play a special role in our approach. Due to the large number of variables and increased number of material constants over the isotropic case, computers are ideally suited to provide designers with timely information. The worksheet, data base and graphics capabilities of personal computers can be utilized to make designing with composites easy, fast and rewarding. We encourage full utilization of whatever computer that is available to the user.

It was not very long ago, in fact in 1976, we were using Texas Instruments SR-52 for our laminate calculations. We were attracted to the programmable calculators by their easy and consistency of operation because we could barely do a 3x3 matrix inversion by hand without errors. The chronology of our calculator/computer experience can be briefly summarized as follows. To us the progress in calculator/computer has been incredible. Designing with composites will continue to progress as computer technology progresses.

Year	Instrument	Memory/steps	Recorder	Cost
1976	TI SR-52	20/120	Mag cards	$300
1978	TI-59	60/480	Mag cards	$300
1980	Sharp 1500	2k RAM/Basic	Cassette	$700
1982	Sinclair	4k RAM/Basic	Cassette	$200
1983	Apple II	64k RAM/Basic	Floppy disk	$3000
1984	IBM PC	128k RAM	Floppy disk	$4000
1984	Macintosh	128k RAM	Micro disk	$3000

We want to fully utilize the opportunities offered by modern computers. Like computers we see unlimited growth in our technology. As we learn to design with composites we will be able to capitalize the superior properties and to reduce cost at the same time. We can have the best of both worlds.

Section 2

CONTRACTED NOTATION

Contracted notation is a simplification of the usual tensorial notation. Instead of having the same number of indices to match the rank of the tensor, like having 2 indices for the second-rank tensor, and 4 for the 4th-rank, the contracted notation reduces the indices by one half. Single index will suffice for the second-rank tensors; double indices, for the 4th-rank. Contracted notation cannot be applied to odd-rank tensors.

It is also a recommended practice to use engineering shear strain, in place of the tensorial shear strain when contracted notation is used. Thus the factor of two must be properly and consistently applied. (Twisting curvature should also be of the engineering rather than tensorial type.) The components of the compliance must also be corrected in addition to thr contraction of the indices. The numeric factors of 1, 2 and 4 when incorrectly or inconsistently applied can be a source of unnecessary complication (leading to unsymmetric stiffness matrix) and confusion.

2.1 CONTRACTED STRESS - This contraction is straightforward. No numeric correction is necessary. There are two systems of notations for the stress components; viz., the letter and numeric subscripts.

TABLE 2.1 CONTRACTION OF STRESS COMPONENTS

Regular letter subs	σ_{xx}	σ_{yy}	σ_{zz}	σ_{yz}	σ_{zx}	σ_{xy}
Regular numeric subs	σ_{11}	σ_{22}	σ_{33}	σ_{23}	σ_{31}	σ_{12}
Contracted numeric	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
Contracted letter	σ_x	σ_y	σ_z	σ_t	σ_u	σ_s

The contraction of the normal stress components is natural and well accepted but that for the shear is not universally accepted. Our contraction of the numeric subscripts is more popular because it follows the same order of 1-2-3 if the plane of the shear stress is designated by its normal; e.g., the 2-3 plane by 1. The contraction of the letter subscripts is arbitrary but consistent with the purpose of contraction. Mixing of single and double subscripts is not recommended.

2.2 CONTRACTED STRAIN

- This contraction needs a numeric correction of 2 for the shear components because engineering shear is used. The usual definition of the tensorial strain-displacement relation is:

$$\epsilon_{ij} = (u_{i,j}+u_{j,i})/2 \tag{2.1}$$

The definition of engineering shear strain is:

$$\epsilon_6 = 2\epsilon_{12} = \partial u/\partial y+\partial v/\partial x,\ \epsilon_5 = 2\epsilon_{31},\ \epsilon_4 = 2\epsilon_{23}. \tag{2.2}$$

The strain contraction is summarized as follows:

TABLE 2.2 CONTRACTION OF STRAIN COMPONENTS

Regular letter subs	ϵ_{xx}	ϵ_{yy}	ϵ_{zz}	ϵ_{yz}	ϵ_{zx}	ϵ_{xy}
Regular numeric subs	ϵ_{11}	ϵ_{22}	ϵ_{33}	ϵ_{23}	ϵ_{31}	ϵ_{12}
Contracted numeric	ϵ_1	ϵ_2	ϵ_3	$\epsilon_4/2$	$\epsilon_5/2$	$\epsilon_6/2$
Contracted letter	ϵ_x	ϵ_y	ϵ_z	$\epsilon_t/2$	$\epsilon_u/2$	$\epsilon_s/2$

2.3 CONTRACTED STIFFNESS

- The stiffness matrix for the generalized Hooke's law in its uncontracted form is:

$$\{\sigma\} = [C]\{\epsilon\},\ \text{or}\ \sigma_{ij} = C_{ijkl}\epsilon_{kl},\ i,j,k,l = 1,2,3 \tag{2.3}$$

This can be represented by a matrix multiplication table, in Table 2.3.

Since both stress and strain are symmetric; i.e.,

$$\sigma_{ij} = \sigma_{ji},\ \epsilon_{ij} = \epsilon_{ji} \tag{2.4}$$

a typical row in this table will becomes:

$$\sigma_{11} = C_{1111}\epsilon_{11}+C_{1122}\epsilon_{22}+C_{1133}\epsilon_{33}+(C_{1123}+C_{1132})\epsilon_{23}$$

$$+(C_{1131}+C_{1113})\epsilon_{31}+(C_{1112}+C_{1121})\epsilon_{12} \quad (2.5)$$

TABLE 2.3 GENERALIZED HOOKE'S LAW IN UNCONDENSED FORM

	ϵ_{11}	ϵ_{22}	ϵ_{33}	ϵ_{23}	ϵ_{32}	ϵ_{31}	ϵ_{13}	ϵ_{12}	ϵ_{21}
σ_{11}	C_{1111}	C_{1122}	C_{1133}	C_{1123}	C_{1132}	C_{1131}	C_{1113}	C_{1112}	C_{1121}
σ_{22}	C_{2211}	C_{2222}	C_{2233}	C_{2223}	C_{2232}	C_{2231}	C_{2213}	C_{2212}	C_{2221}
-									
-									
-									
σ_{21}	C_{2111}	C_{2122}	C_{2133}	C_{2123}	C_{2132}	C_{2131}	C_{2113}	C_{2112}	C_{2121}

If we now introduce the engineering shear strain and the contracted notation for all the indices, we have:

$$\sigma_1 = C_{11}\epsilon_1+C_{12}\epsilon_2+C_{13}\epsilon_3+C_{14}\epsilon_4+C_{15}\epsilon_5+C_{16}\epsilon_6 \quad (2.6)$$

In summary, the generalized Hooke's law in contracted notation is:

$$\{\sigma\} = [C]\{\epsilon\}, \text{ or } \sigma_i = C_{ij}\epsilon_j,\ i,j = 1,2,3,4,5,6 \quad (2.7)$$

Thus the indices for the stiffness components follow precisely those for the contraction of stress. No correction factor for the contraction. This easy conversion from four to two indices is made possible by having:

- Symmetry of stress and strain,
- Symmetry of the stiffness matrix, and
- Use of engineering shear strain.

2.4 CONTRACTED COMPLIANCE

- The generalized Hooke's law in terms of compliance is:

$$\{\epsilon\} = [S]\{\sigma\}, \text{ or } \epsilon_{ij} = S_{ijkl}\sigma_{kl},\ i,j,k,l = 1,2,3 \tag{2.8}$$

The first and the eighth rows for this uncondensed, uncontracted form of the generalized Hooke's law are:

$$\begin{aligned}\epsilon_{11} = {} & S_{1111}\sigma_{11}+S_{1122}\sigma_{22}+S_{1133}\sigma_{33}+S_{1123}\sigma_{23}+S_{1132}\sigma_{32}\\ & +S_{1131}\sigma_{31}+S_{1113}\sigma_{13}+S_{1112}\sigma_{12}+S_{1121}\sigma_{21}\end{aligned} \tag{2.9}$$

$$\begin{aligned}\epsilon_{12} = {} & S_{1211}\sigma_{11}+S_{1222}\sigma_{22}+S_{1233}\sigma_{33}+S_{1223}\sigma_{23}+S_{1232}\sigma_{32}\\ & +S_{1231}\sigma_{31}+S_{1213}\sigma_{13}+S_{1212}\sigma_{12}+S_{1221}\sigma_{21}\end{aligned} \tag{2.10}$$

If we now apply the contracted notation of stress and strain (with engineering shear), and assume symmetry for the compliance matrix, we will have:

$$\{\epsilon\} = [S]\{\sigma\}, \text{ or } \epsilon_i = S_{ij}\sigma_j,\ i,j = 1,2,3,4,5,6 \tag{2.11}$$

But the contraction of the compliance matrix requires additional numeric corrections. The contracted forms for Equations 2.8 and 2.9 are:

$$\epsilon_1 = S_{11}\sigma_1+S_{12}\sigma_2+S_{13}\sigma_3+S_{14}\sigma_4+S_{15}\sigma_5+S_{16}\sigma_6 \tag{2.12}$$

$$\epsilon_6 = S_{61}\sigma_1+S_{62}\sigma_2+S_{63}\sigma_3+S_{64}\sigma_4+S_{65}\sigma_5+S_{66}\sigma_6 \tag{2.13}$$

If we compare term by term between Equations 2.12 and 2.9, and 2.13 and 2.10, we will arrive at the following numeric corrections by factors of 1, 2 and 4:

$$\begin{aligned}& S_{11} = S_{1111}, \ldots S_{12} = S_{1122}, \ldots\\ & S_{14} = 2S_{1123}, S_{15} = 2S_{1131}, \ldots S_{61} = 2S_{1211}, \ldots\\ & S_{44} = 4S_{2323}, S_{45} = 4S_{2331}, \ldots\end{aligned} \tag{2.14}$$

Thus the contracted compliance matrix can be viewed as having four equal 3x3 sub-matrices. The correction factor is unity for the upper-left sub-matrix; 2, for the lower-left, and upper-right; and 4, for the lower-right. These correction factors are necessary and are the results of the symmetry of stress and strain, the symmetry of the compliance matrix, and the use of engineering shear.

If engineering shear is not used, a factor of 2 must be applied to the last three columns of the stiffness matrix and the last three rows of the compliance matrix. These matrices are no longer symmetric.

2.5 CONCLUSIONS - The use of contracted notation reduces the number of indices but must be applied consistently. Mixing of single and double indices such as the use of double index for shear stress and shear strain is not recommended. Furthermore, engineering shear strain is recommended for the contracted notation (Table 2.2). While the contracted stiffness matrix is derived from the uncontracted without correction factors (Table 2.3), the contracted compliance matrix requires correction factors of 1, 2 and 4 (Equation 2.14).

Section 3

GENERALIZED HOOKE'S LAW

Generalized Hooke's law is a linear stress-strain relation for anisotropic materials. Its derivation from the energy consideration is a basic postulate in the theory of elasticity. We will use contracted notation described in the last section.

3.1 MATRIX MULTIPLICATION TABLES – It is convenient to represent a matrix multiplication, such as:

$$\{\sigma\} = [C]\{\epsilon\} \quad \text{or} \quad \sigma_i = C_{ij}\epsilon_j,\ i,j = 1,2,3,4,5,6 \qquad (3.1)$$

by the following matrix multiplication table:

TABLE 3.1 GENERALIZED HOOKE'S LAW IN TERMS OF STIFFNESS

	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6
σ_1	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}
σ_2	C_{21}	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}
σ_3	C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	C_{36}
σ_4	C_{41}	C_{42}	C_{43}	C_{44}	C_{45}	C_{46}
σ_5	C_{51}	C_{52}	C_{53}	C_{54}	C_{55}	C_{56}
σ_6	C_{61}	C_{62}	C_{63}	C_{64}	C_{65}	C_{66}

This is the most general law for an anisotropic linearly elastic material. There are 36 components or constants for this materials. This material has no material symmetry and is also called the triclinic material. This stiffness matrix however is symmetric from energy consideration; i.e.,

$$C_{ij} = C_{ji} \tag{3.2}$$

With this symmetry or reciprocal relation only 21 of the 36 constants are independent.

3.2 MATERIAL SYMMETRIES

– If any material symmetry exists, the number of constants will reduce. For example, if the 1-2 plane or z=0 is a plane of symmetry, all constants associated with the positive 3- or z-axis must be the same as those with the negative 3- or z-axis. Shear strain components ϵ_4 and ϵ_5, or ϵ_{yz} and ϵ_{xz} in the uncontracted notation, respectively, are coupled only with shear stress components σ_4 and σ_5. The Hooke's law in Table 3.1 must be simplified for a material symmetric with respect to the z=0 plane or a monoclinic material, as follows:

TABLE 3.2 STIFFNESS OF A MONOCLINIC (Z=0) MATERIAL

	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6
σ_1	C_{11}	C_{12}	C_{13}	0	0	C_{16}
σ_2	C_{21}	C_{22}	C_{23}	0	0	C_{26}
σ_3	C_{31}	C_{32}	C_{33}	0	0	C_{36}
σ_4	0	0	0	C_{44}	C_{45}	0
σ_5	0	0	0	C_{54}	C_{55}	0
σ_6	C_{61}	C_{62}	C_{63}	0	0	C_{66}

As expressed in this coordinate system, there are 20 constants, of which 13 are independent. If the coordinate system changes, for example, to an arbitrary plane of symmetry, the number of nonzero constants will increase up to 36. But the independent constants remain 13, which is the case of a monoclinic material for all coordinate systems.

As the level of material symmetry increases, the number of independent constants reduces. If we have symmetry in three orthogonal planes we have an orthotropic material. The number of independent constants are now 9. If the planes of symmetry coincide with the reference coordinate system, the nonzero components are 12; this is shown in Table 3.3. If the symmetry planes are not coincident with the reference coordinates, the nonzero components will be those shown in Table 3.1. If one of the symmetry planes coincide with one reference coordinate, the nonzero components will be those shown in Table 3.2. The number of independent constants remain 9 for orthotropic materials irrespective of the orientation of the symmetry planes.

TABLE 3.3 STIFFNESS OF AN ORTHOTROPIC MATERIAL

	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6
σ_1	C_{11}	C_{12}	C_{13}	0	0	0
σ_2	C_{21}	C_{22}	C_{23}	0	0	0
σ_3	C_{31}	C_{32}	C_{33}	0	0	0
σ_4	0	0	0	C_{44}	0	0
σ_5	0	0	0	0	C_{55}	0
σ_6	0	0	0	0	0	C_{66}

The next level of material symmetry is the transversely isotropic material, which has 5 independent constants. If the isotropic plane coincides with one of the planes of the coordinate system, the nonzero components are 9; this is shown in Table 3.4.

Finally the isotropic material has 2 independent constants for all coordinate systems. The nonzero components remain 9. This is shown in Table 3.4.

Because of transverse isotropy, the following four relations were used in reducing the number of independent constants from 9 for the orthotropic material to 5 for the transversely isotropic material:

TABLE 3.4 STIFFNESS OF AN TRANSVERSELY ISOTROPIC (x=constant) MATERIAL

	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6
σ_1	C_{11}	C_{12}	C_{12}	0	0	0
σ_2	C_{21}	C_{22}	C_{23}	0	0	0
σ_3	C_{21}	C_{32}	C_{22}	0	0	0
σ_4	0	0	0	$(C_{22}-C_{23})/2$	0	0
σ_5	0	0	0	0	C_{66}	0
σ_6	0	0	0	0	0	C_{66}

$$C_{22} = C_{33}, \quad C_{13} = C_{12},$$
$$C_{55} = C_{66}, \quad C_{44} = (C_{22}-C_{23})/2 \tag{3.3}$$

Since the yz-plane is isotropic, coordinates y and z are equal and interchangeable; i.e., indices 2 equal to 3, and 5 equal to 6 in the contracted notation. The last of Equation 3.3 is derived from the equivalence between pure shear, and the combined tension and compression.

If a material is fully isotropic the number of independent constants reduce from 5 to 2 resulting from the following three relations:

$$C_{11} = C_{22} = C_{33}, \; C_{12} = C_{23} = C_{31},$$
$$C_{44} = C_{55} = C_{66} = (C_{11}-C_{12})/2 \tag{3.4}$$

We have shown the stiffness components are functions of material symmetries. The compliance components follow the same pattern of the nonzero and the number of independent components. They have the same appearance as the stiffness components in Table 3.1 to 3.4. We merely replace C_{ij} by S_{ij}.

There is however one exception; i.e., the equivalence of pure shear, and the combined tension and compression applied at a 45-degree orientation would lead to the following relation for the compliance components:

$$S_{44} = 2(S_{22}-S_{23}),\ S_{55} = 2(S_{33}-S_{31}),$$
$$S_{66} = 2(S_{11}-S_{12}) \tag{3.5}$$

In terms of engineering constants, we can express the equivalence as:

$$G = E/2(1+\nu) \tag{3.6}$$

where G, E, ν are the shear and Young's modulus, and the Poisson's ratio.

We will now present a summary of the Hooke's law in Table 3.5. The on-axis refers to the symmetry axes; the off-axis, the rotation about one of the reference axes; and the general, rotation about any axis.

TABLE 3.5 SUMMARY OF 3-DIMENSIONAL MATERIAL SYMMETRIES

Type of Material Symmetry	Number of Independent Constants	Number of Nonzero: On-axis	Number of Nonzero: Off-axis	Number of Nonzero: General
Triclinic	21	36	36	36
Monoclinic	13	20	36	36
Orthotropic	9	12	20	36
Trans. Isotropic	5	12	20	36
Isotropic	2	12	12	12

The behavior of an anisotropic material depends not as much on the number of independent constants as the nonzero components. For example, the on-axis orthotropic and the on-axis transversely isotropic materials behave the same qualitatively as an isotropic material. They all have 12 nonzero components, and geometrically arranged like those in Table 3.3 or 3.4. For these materials, the shear and normal components of stress and strain are not coupled. When an orthotropic or transversely isotropic material rotates away from its symmetry axes about one reference axis, this off-axis orientation results in 20 nonzero components. Now shear coupling is present and this material will behave similar to a monoclinic material in its on-axis orientation. If the orthotropic or transversely isotropic material rotates about an axis other than the three reference axes, the nonzero components are now 36 and will behave the same as a triclinic material shown in Table 3.1.

3.3 ENGINEERING CONSTANTS

- The definitions of the Young's of an anisotropic material is straightforward, as follows:

$$E_1 = 1/S_{11},\ E_2 = 1/S_{22},\ E_3 = 1/S_{33} \tag{3.7}$$

But the definition of shear moduli is arbitrary. We show two definitions; using one or two subscripts:

$$E_4 = G_{23} = 1/S_{44},\ E_5 = G_{31} = 1/S_{55},\ E_6 = G_{12} = 1/S_{66} \tag{3.8}$$

We prefer the single subscript definition because it is consistent with the intent of the contracted notation.

The definitions of the engineering constants for the Poisson, shear coupling, and others are even less standardized and are in fact conflicting. We will show both definitions for a monoclinic or off-axis orthotropic or transversely isotropic material, in the following tables. The difference lies in the normalizing factor, by using the same for each column or row.

In Table 3.6 each column is normalized by the same engineering constant. We prefer this definition because the interpretation of simple tests such as an uniaxial test can be readily done. But many authors use the row normalization for the definition of engineering constants. This is shown in Table 3.7.

Thus the definition of the coupling coefficients ν_{ij} depends how the normalization is applied. In Table 3.7 column normalization is done. We have for the case of the larger Young's modulus in E_1 than in E_2:

ν_{21} = major Poisson's ratio

= longitudinal Poisson's ratio

= the larger of the Poisson's ratio

$$= -S_{12}/S_{11} \tag{3.9}$$

TABLE 3.6 COLUMN-NORMALIZED ENGINEERING CONSTANTS OF A MONOCLINIC MATERIAL

	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
ϵ_1	$1/E_1$	$-\nu_{12}/E_2$	$-\nu_{13}/E_3$	0	0	ν_{16}/E_6
ϵ_2	$-\nu_{21}/E_1$	$1/E_2$	$-\nu_{23}/E_3$	0	0	ν_{26}/E_6
ϵ_3	$-\nu_{31}/E_1$	$-\nu_{32}/E_2$	$1/E_3$	0	0	ν_{36}/E_6
ϵ_4	0	0	0	$1/E_4$	ν_{45}/E_5	0
ϵ_5	0	0	0	ν_{54}/E_4	$1/E_5$	0
ϵ_6	ν_{61}/E_1	ν_{62}/E_2	ν_{63}/E_3	0	0	$1/E_6$

TABLE 3.7 ROW-NORMALIZED ENGINEERING CONSTANTS OF A MONOCLINIC MATERIAL

	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
ϵ_1	$1/E_1$	$-\nu_{12}/E_1$	$-\nu_{13}/E_1$	0	0	ν_{16}/E_1
ϵ_2	$-\nu_{21}/E_2$	$1/E_2$	$-\nu_{23}/E_2$	0	0	ν_{26}/E_2
..						..
..						..
ϵ_6	ν_{61}/E_6	ν_{62}/E_6	ν_{63}/E_6	0	0	$1/E_6$

ν_{12} = minor Poisson's ratio

$$= \nu_{21}E_2/E_1$$

$$= -S_{12}/S_{22} \qquad (3.10)$$

In Table 3.7 where the row normalization is used, we have the definitions for the coupling coefficients defined just the opposite of those in Equation 3.9; where again we assume E_1 larger than E_2.

ν_{12} = major Poisson's ratio

= the larger of the Poisson's ratio

$= -S_{12}/S_{22}$ (3.11)

ν_{21} = minor Poisson's ratio

$= \nu_{12}E_2/E_1$

$= -S_{12}/S_{11}$ (3.12)

As indicated earlier, we do not recommend the coupling coefficients by row normalization shown in Table 3.7.

3.4 CONCLUSIONS - Generalized Hooke's law in three dimensions can be simplified by the presence of material symmetry and the chosen orientation of the reference coordinates. The nature of materials behavior however is dictated more by the number of nonzero components than by the number of independent constants. Care must be exercised when we use the engineering constants of the coupling coefficients. Both definitions, shown in Equations 3.9 to 3.12, are used in the literature.

Section 4

PLANE STRESS & PLANE STRAIN

Stress analysis is done in two dimensions for most materials including composites. The generalized Hooke's law is much simplified for the two dimensional cases of plane stress and plane strain.

4.1 PLANE STRESS - The assumed zero and nonzero stress and strain components for various anisotropic materials are listed in Table 4.1. The term on- or off- refer to the coordinate axes placed on or off the material symmetry axes, respectively. The off axis is a rotation about one of the reference coordinate axes.

TABLE 4.1 STRESS AND STRAIN COMPONENTS UNDER PLANE STRESS

Material Symmetry	No symmetry Triclinic Off-monoclinic	On-monoclinic Off-ortho Off-trans-iso	On-ortho On-trans-iso Isotropic
Nonzero Constants	36	20	9
$\sigma_1,\sigma_2,\sigma_6$	$\neq 0$	$\neq 0$	$\neq 0$
$\sigma_3,\sigma_4,\sigma_5$	$=0$	$=0$	$=0$
$\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_6$	$\neq 0$	$\neq 0$	$\neq 0$
ϵ_4,ϵ_5	$\neq 0$	$=0$	$=0$

It is assumed that the 1-2 plane is the plane of interest, and the nonzero stress and strain components in this plane can be related by a specialized

Hooke's law, as follows:

$$\{\epsilon\} = [S]\{\sigma\}, \text{ or } \epsilon_i = S_{ij}\sigma_j, \ i,j = 1,2,6 \tag{4.1}$$

where the compliance for plane stress has the same components as that for the 3-dimensional Hooke's law because stress components are specified. The other nonzero component for all materials is the normal strain in the thickness or the 3 direction. From the specialized Hooke's law we can show:

$$\epsilon_3 = S_{31}\sigma_1 + S_{32}\sigma_2 + S_{36}\sigma_6 \tag{4.2}$$

where the first two compliance components are Poisson couplings; and the last component, shear coupling. If the material is isotropic, the two Poisson couplings are equal and shear coupling vanishes:

$$\epsilon_3 = -\nu(\sigma_1+\sigma_2)/E \tag{4.3}$$

The 3-dimensional stress-strain relations for the case of plane stress in terms of stiffness matrix for materials with 20 or less nonzero components, shown for example in Table 3.2, are as follows:

$$\begin{aligned}
\sigma_1 &= C_{11}\epsilon_1 + C_{12}\epsilon_2 + C_{13}\epsilon_3 + C_{16}\epsilon_6 \\
\sigma_2 &= C_{21}\epsilon_1 + C_{22}\epsilon_2 + C_{23}\epsilon_3 + C_{26}\epsilon_6 \\
\sigma_6 &= C_{61}\epsilon_1 + C_{62}\epsilon_2 + C_{63}\epsilon_3 + C_{66}\epsilon_6 \\
\sigma_3 &= C_{31}\epsilon_1 + C_{32}\epsilon_2 + C_{33}\epsilon_3 + C_{36}\epsilon_6 = 0 \\
\sigma_4 &= \sigma_5 = 0
\end{aligned} \tag{4.4}$$

We can eliminate the normal strain in the 3-direction by solving the fourth equation in (4.4):

$$\epsilon_3 = -(C_{31}\epsilon_1 + C_{32}\epsilon_2 + C_{36}\epsilon_6)/C_{33} \tag{4.5}$$

By substituting Equation (4.5) into (4.4), we now have the stress-strain relation for a plane stress in the 1-2 plane in terms of reduced stiffness:

$$\{\sigma\} = [Q]\{\epsilon\}, \text{ or } \sigma_i = Q_{ij}\epsilon_j, \ i,j = 1,2,6 \tag{4.6}$$

where $Q_{ij} = C_{ij} - C_{i3}C_{j3}/C_{33}$ (4.7)

= reduced stiffness for plane stress

If the plane stress is in the 3-1 plane:

$$Q_{ij} = C_{ij} - C_{i2}C_{j2}/C_{22},\ i,j = 1,3,5 \quad (4.8)$$

The number of independent and nonzero constants for each symmetry will be listed in Table 4.2.

TABLE 4.2 ELASTIC MODULI UNDER PLANE STRESS

Material Symmetry (independent constants)	Anisotropic(6) Off-orthotropic(4) Off-square symm(3)	On-orthotropic(4) On-square symm(3) Isotropic(2)
Nonzero Components	6	4
$Q_{11}, Q_{22}, Q_{12}, Q_{66}$ $S_{11}, S_{22}, S_{12}, S_{66}$	$\neq 0$	$\neq 0$
Q_{16}, Q_{26} S_{16}, S_{26}	$\neq 0$	$= 0$

The behavior of materials is controlled more by the number of nonzero constants than by the number of independent constants. Thus for plane stress, the principal difference between the 6-constant materi' and the 4-constant one is the existence or absence of shear coupling. The square symmetric material has equal stiffness on its symmetry axes, but unlike the isotropic material the in-plane shear is independent. So there are 3 independent constants. A fabric with balanced weave is a square symmetric material.

Engineering constants are defined from the components of compliance:

$$E_1 = 1/S_{11},\ E_2 = 1/S_{22},\ E_6 = G_{12} = 1/S_{66}, \quad (4.9)$$

$$\nu_{21} = -S_{21}/S_{11},\ \nu_{12} = -S_{12}/S_{22} = \nu_{21}E_2/E_1$$

$$\nu_{61} = S_{61}/S_{11},\ \nu_{16} = S_{16}/S_{66} = \nu_{61}E_6/E_1 \qquad (4.10)$$

$$\nu_{62} = S_{62}/S_{22},\ \nu_{26} = S_{26}/S_{66} = \nu_{62}E_6/E_2$$

The coupling coefficients above are defined using normalization by columns as in Table 3.6, not by rows as in Table 3.7. Finally explicit relations between engineering constants and the stiffness components exist through the inversion of the compliance matrix. But these relations are simple only for the on-axis orthotropic material. Simple relations for anisotropic and off-axis orthotropic materials do not exist; i.e.,

$$Q_{11} \neq E_1/(1-\nu_{21}\nu_{12}),\ Q_{22} \neq E_2/(1-\nu_{21}\nu_{12}),$$
$$\qquad (4.11)$$
$$Q_{12} \neq \nu_{21}Q_{11} \neq \nu_{12}Q_{22},\ Q_{66} \neq E_6 \neq G_{12}$$

In order to avoid confusion we designate the on-axis orthotropic constants by letter subscripts to distinguish them from numeric subscripts of the anisotropic and off-axis orthotropic constants:

$$Q_{xx} = E_x/(1-\nu_x\nu_y),\ Q_{yy} = E_y/(1-\nu_x\nu_y),$$
$$\qquad (4.12)$$
$$Q_{xy} = \nu_x Q_{yy} = \nu_y Q_{xx},\ Q_{ss} = E_s$$

For the on-axis square symmetric material:

$$Q_{xx} = Q_{yy} = E_x/(1-\nu_x{}^2),\ Q_{xy} = \nu_x E_x/(1-\nu_x{}^2),$$
$$\qquad (4.13)$$
$$Q_{ss} = E_s,\ E_s \neq E_x/2(1+\nu_x)$$

where the shear modulus is not dependent on the Young's modulus and Poisson's ratio, making this a 3-constant material. These constants are all defined relative to the symmetry axes. There are two sets of such axes, one 45 degree from the other.

For the isotropic material:

$$Q_{xx} = Q_{yy} = E/(1-\nu^2),\ Q_{xy} = \nu E/(1-\nu^2),$$
$$\qquad (4.14)$$
$$Q_{ss} = G = E/2(1+\nu)$$

The last equation is derived from that from Equation 3.3:

$$G = (Q_{xx}-Q_{xy})/2 \tag{4.15}$$

The basic differences between the square symmetric versus isotropic materials are: the 3 versus 2 independent constants, and that the engineering constants must be measured from the symmetry axes versus those from any axes, respectively. The subscripts in the engineering constants in Equation 4.13 are there to signify nonisotropic constants.

4.2 PLANE STRAIN - This is the other 2-dimensional state very analogous to the state of plane stress.

TABLE 4.3 STRESS AND STRAIN COMPONENTS UNDER PLANE STRAIN

Material Symmetry	No symmetry Triclinic Off-monoclinic	On-monoclinic Off-ortho Off-trans-iso	On-ortho On-trans-iso Isotropic
Nonzero Constants	36	20	9
$\epsilon_1,\epsilon_2,\epsilon_6$	$\neq 0$	$\neq 0$	$\neq 0$
$\epsilon_3,\epsilon_4,\epsilon_5$	$=0$	$=0$	$=0$
$\sigma_1,\sigma_2,\sigma_3,\sigma_6$	$\neq 0$	$\neq 0$	$\neq 0$
σ_4,σ_5	$\neq 0$	$=0$	$=0$

Since strain components are specified in plane strain, the stiffness components for 2-dimensional plane strain are the same as those for the 3-dimensional Hooke's law. Therefore, the stress and strain components in the 1-2 plane, for example, are related as follows:

$$\{\sigma\} = [C]\{\epsilon\}, \text{ or } \sigma_i = C_{ij}\epsilon_j,\ i,j = 1,2,6 \tag{4.16}$$

For materials other than the triclinic, 21-constants type, we can show that the stress component normal to the 1-2 plane can be determined:

$$\sigma_3 = C_{31}\epsilon_1 + C_{32}\epsilon_2 + C_{36}\epsilon_6 \tag{4.17}$$

The compliance components, however, must be modified for the plane strain state. This modification can be derived by substituting Equation 4.15 into the generalized Hooke's law to eliminate σ_3 as an independent variable. This derivation is analogous to the reduced stiffness matrix for the plane stress case. Now we have the reduced compliance case for the plane strain. Assuming that the 1-2 plane is the plane strain, we have

$$\{\sigma\} = [R]\{\epsilon\}, \text{ or } \sigma_i = R_{ij}\epsilon_j,\ i,j = 1,2,6 \tag{4.18}$$

where $$R_{ij} = S_{ij} - S_{i3}S_{j3}/S_{33} \tag{4.19}$$

= reduced compliance for plane strain

We can derive the engineering constants associated with plane strain same as those with plane stress; e.g.,

$$E_1 = 1/R_{11}, \ldots \nu_{21} = -R_{21}/R_{11}, \ldots \tag{4.20}$$

The coupling coefficients are based on normalization by columns as in Table 3.6 and Equation 4.10. Again we wish to emphasize that there are at least two systems of normalization, one by columns and one by rows. We recommend the former.

The relations between engineering constants and the stiffness components are not as straightforward because matrix inversion is involved.

The independent and nonzero constants of anisotropic materials under plane strain, analogous to those in Table 4.2 for plane stress, are shown in Table 4.4.

4.3 CONCLUSIONS – The stiffness and compliance matrices in the generalized Hooke's law for 3-dimensional stress and strain cannot be transfered directly to plane stress and plane strain. Modifications to the 3-dimensional state are necessary, and summarized in Table 4.5.

TABLE 4.4 ELASTIC MODULI UNDER PLANE STRAIN

Material Symmetry (independent constants)	Anisotropic(6) Off-orthotropic(4) Off-square symm(3)	On-orthotropic(4) On-square symm(3) Isotropic(2)
Nonzero Components	6	4
$C_{11}, C_{22}, C_{12}, C_{66}$ $R_{11}, R_{22}, R_{12}, R_{66}$	$\neq 0$	$\neq 0$
C_{16}, C_{26} R_{16}, R_{26}	$\neq 0$	$= 0$

TABLE 4.5 SUMMARY OF 3- AND 2-DIMENSIONAL ELASTIC MODULI

Dimensions	3-Dimension	2-D Plane Stress	2-D Plane Strain
Stiffness	C_{ij}	Q_{ij}	C_{ij}
Compliance	S_{ij}	S_{ij}	R_{ij}

Section 5

STRESS AND STRAIN TRANSFORMATIONS

5.1 TRANSFORMATION EQUATIONS – Stress and strain at a point are functions of the reference coordinate axes. This coordinate axes dependence is of fundamental importance not only because the true nature of stress and strain is explained, but also the relations between laminate stress and strain and ply stress and strain can be explicitly stated. One outstanding features of composite materials is the highly directionally dependent properties. It is often advantageous to use unidirectional or laminated composites is their off-axis orientations. The transformation equations are essential in the understanding of this off-axis and the principal axis orientations. The latter will be used as one design method of selecting the optimum laminate orientation.

Applied loads to a structure are usually given in the laminate axes while failure criteria for example are usually applied to the stress or strain relative to the ply axes. The transformation equations allow us to go from one coordinate system to another.

The components of stress and strain change in accordance with specific transformation equations. The equations for stress are different from those for strain because we have elected to use the contracted notation which requires the use of engineering shear strain. We have shown the transformation equations in both the matrix and index notations, as follows:

$$\begin{aligned}
&\{\sigma'\} = [J]\{\sigma\} && \sigma_i' = J_{ij}\sigma_j,\ i,j = 1,2,6 \\
&\{\sigma\} = [J]^{-1}\{\sigma'\} && \sigma_i = J_{ij}^{-1}\sigma_j' \\
&\{\epsilon\} = [J^T]\{\epsilon'\} && \epsilon_i = J_{ji}\epsilon_j' \qquad (5.1) \\
&\{\epsilon'\} = [J^T]^{-1}\{\epsilon\} && \epsilon_i' = J_{ji}^{-1}\epsilon_i
\end{aligned}$$

where

$$[J] = \begin{vmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{vmatrix} \tag{5.2}$$

$$[J]^{-1} = \begin{vmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2-n^2 \end{vmatrix} \tag{5.3}$$

$$[J^T] = \begin{vmatrix} m^2 & n^2 & -mn \\ n^2 & m^2 & mn \\ 2mn & -2mn & m^2-n^2 \end{vmatrix} \tag{5.4}$$

$$[J^T]^{-1} = \begin{vmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2-n^2 \end{vmatrix} \tag{5.5}$$

$m = \cos\theta$, $n = \sin\theta$

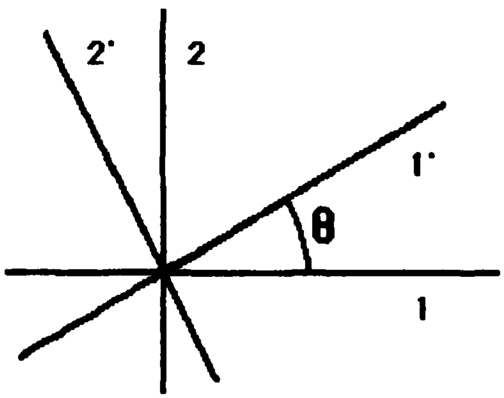


FIGURE 5.1 RELATION BETWEEN THE 1-2 AND 1'-2' COORDINATES.

Typical values for transformation matrices of frequently used ply angles of 0, 90, 45 and -45 degrees are:

$$[J]^{(0)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [J^T]^{-1(0)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.6}$$

$$[J]^{(90)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad [J^T]^{-1(90)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{5.7}$$

$$[J]^{(45)} = \begin{bmatrix} .5 & .5 & 1 \\ .5 & .5 & -1 \\ -.5 & .5 & 0 \end{bmatrix} \qquad [J^T]^{-1(45)} = \begin{bmatrix} .5 & .5 & .5 \\ .5 & .5 & -.5 \\ -1 & 1 & 0 \end{bmatrix} \tag{5.8}$$

$$[J]^{(-45)} = \begin{bmatrix} .5 & .5 & -1 \\ .5 & .5 & 1 \\ .5 & -.5 & 0 \end{bmatrix} \qquad [J^T]^{-1(-45)} = \begin{bmatrix} .5 & .5 & -.5 \\ .5 & .5 & .5 \\ 1 & -1 & 0 \end{bmatrix} \tag{5.9}$$

5.2 MULTIPLE-ANGLE TRANSFORMATION

- This is an alternative form using linear combinations of the stress components:

$$p_\sigma = (\sigma_1+\sigma_2)/2,\ q_\sigma = (\sigma_1-\sigma_2)/2,\ r_\sigma = \sigma_6 \tag{5.10}$$

For the strain components, we have analogous combinations except that tensorial strain must be used. The factor of 1/2 is needed.

$$p_\epsilon = (\epsilon_1+\epsilon_2)/2,\ q_\epsilon = (\epsilon_1-\epsilon_2)/2,\ r_\epsilon = \epsilon_6/2 \tag{5.11}$$

We can rewrite the stress transformation equations in matrix form using the linear combinations in Equation 5.10:

$$\begin{gathered} \{p',q',r'\}_\sigma = [K]\{p,q,r\}_\sigma \\ \{p,q,r\}_\sigma = [K]^{-1}\{p',q',r'\}_\sigma \end{gathered} \tag{5.12}$$

Similarly, we can write the strain transformation in matrix form:

$$\{p',q',r'\}_\varepsilon = [K^T]\{p,q,r\}_\varepsilon$$

$$\{p',q',r'\}_\varepsilon = [K^T]^{-1}\{p,q,r\}_\varepsilon \qquad (5.13)$$

where $$[K] = [K^T]^{-1} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta \\ 0 & -\sin 2\theta & \cos 2\theta \end{vmatrix} \qquad (5.14)$$

$$[K]^{-1} = [K^T] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{vmatrix} \qquad (5.15)$$

These transformation is simpler than those with the second power of sines and cosines because the matrices are rigid-body rotations with double angles. In fact, Mohr's circles are constructed by this double angle rotation. Stress and strain now have the same transformation because tensorial shear strain is used in Equation 5.11.

5.3 PRINCIPAL STRESS AND STRAIN - For every state of stress, there is one particular orientation of the coordinate axes when the normal components reach extremum values and the shear vanish. Using Equation 5.12, this orientation θ_o is found by letting:

$$r' = 0 = -q\sin 2\theta_o + r\cos 2\theta_o, \text{ or}$$

$$\tan 2\theta_o = q/r = 2\sigma_6/(\sigma_1 - \sigma_2) \qquad (5.16)$$

$$\sigma_1^{prin} = \sigma_I = p+R,\ \ \sigma_2^{prin} = \sigma_{II} = p-R,\ R^2 = q^2+r^2,\ \sigma_6^{prin} = 0$$

Subscript σ is left out here; all quantities are related to stress.
At 45 degree from the principal axes, the shear reaches maximum and the normal components are equal.

at $\theta = \theta_o \pm 45$

$$\sigma_1^{max\ shear} = \sigma_2^{max\ shear} = p,\ \sigma_6^{max\ shear} = \pm R \tag{5.17}$$

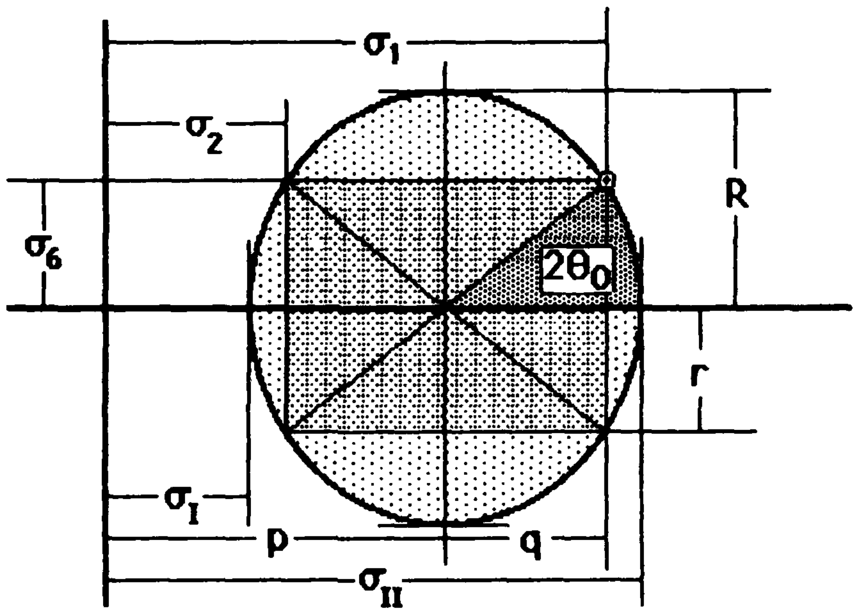


FIGURE 5.2 MOHR'S CIRCLE REPRESENTATION OF STRESS AND ITS PRINCIPAL COMPONENTS.

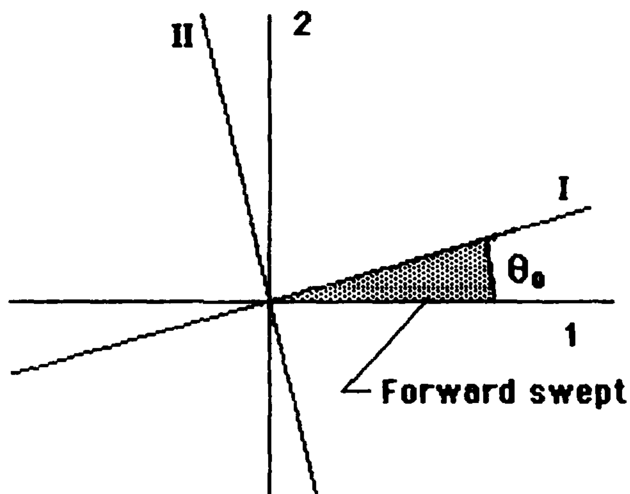


FIGURE 5.3 RELATION BETWEEN 1-2 AND I-II AXES. COUNTER CLOCKWISE PHASE ANGLE FROM 1 TO I IN FIGURE 5.2 IS CLOCKWISE HERE OR FORWARD SWEPT WHEN WE "FLY" ALONG THE 2-AXIS.

Similar relations for the principal strain exist (subscript ϵ is omitted here also):

$r' = 0 = -q\sin 2\theta_o + r\cos 2\theta_o$, or

$$\tan 2\theta_o = r/q = \epsilon_6/(\epsilon_1-\epsilon_2) \qquad (5.18)$$

$$\epsilon_1^{prin} = \epsilon_I = p+R,\ \epsilon_2^{prin} = \epsilon_{II} = p-R,\ R^2 = q^2+r^2,\ \epsilon_6^{prin} = 0$$

at $\theta = \theta_o - 45$

$$\epsilon_1^{max\ shear} = \epsilon_2^{max\ shear} = q_I,\ \epsilon_6^{max\ shear} = -R \qquad (5.19)$$

5.4 NUMERICAL EXAMPLES – Given the following state of stress:

$$\sigma_i = \{100, -30, 50\}. \qquad (5.20)$$

Find the transformed components and the principal and the maximum shear axes: Results are shown in the following charts obtained by substituting the values in Equation 5.20 into the first of Equation 5.1.

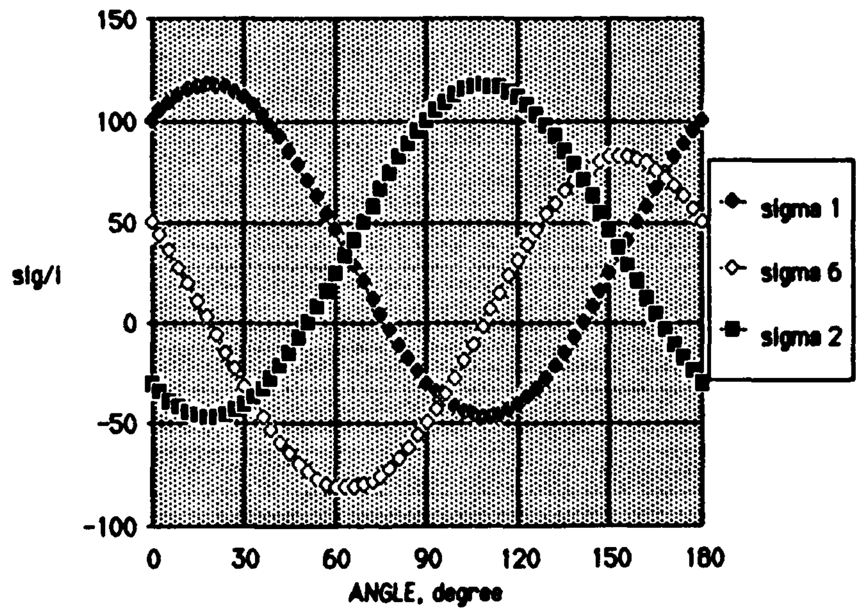


FIGURE 5.4 TRANSFORMED STRESS COMPONENTS OF EQUATION 5.20.

Figure 5.4 shows the variation of the stress components as a function of coordinate axes, as measured by the angle θ in Figure 5.1. The true essence of transformation is the principal axes, which, for exmaple, are analogous to the symmetry axes of materials. From Equation 5.16, the phase angle based on the initial stress in Equation 5.20 is:

$$\theta_0 = [\arctan(r/q)]/2 = [\arctan(2\times 50/130)]/2 = 18.8° \qquad (5.21)$$

The nature of stress transformation from the principal axes is graphically illustrated in Figure 5.5:

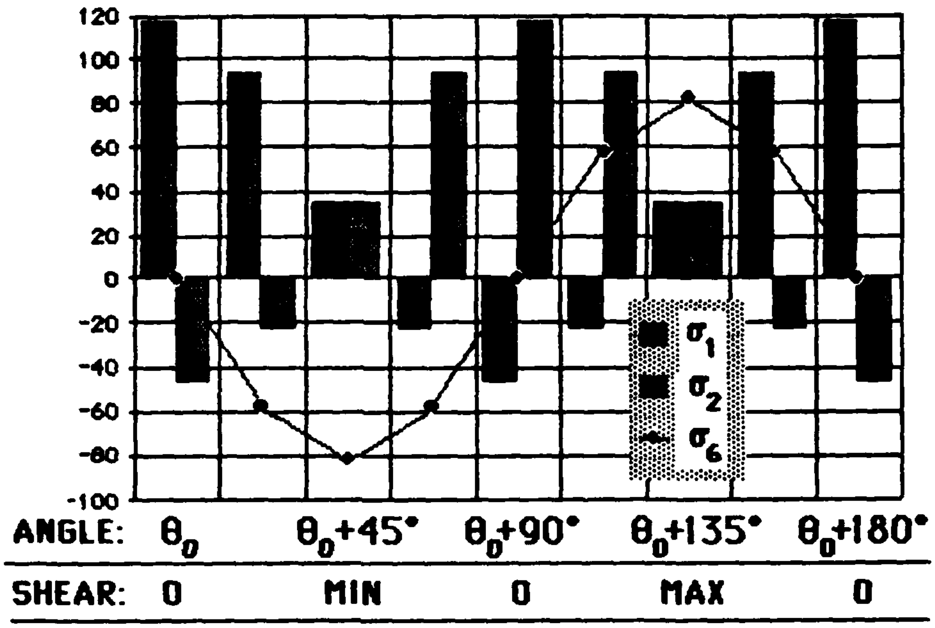


FIGURE 5.5 TRANSFORMED STRESS FROM THE PRINCIPAL AXES.

If we are given the following strain, which has the same numerical values as those for stress in Equation 5.20; i.e.,

$$\epsilon_i = \{100, -30, 50\} \qquad (5.22)$$

What would be the transformed strain components?

We must now use the strain transformation equations in Equation 5.1. Since the transformation equations are different from those for stress, the transformed strain components will be different from those shown in Figure 5.4. We show them in Figure 5.6.

Using Equation 5.18, the phase angle for the principal strain axes are:

$$\theta_0 = [\arctan(50/130)]/2 = 10.5 \text{ degree} \qquad (5.23)$$

Notice that this angle is different from the 18.8 degree for the principal stress in Equation 5.21. This is expected because the transformation equations are difference.

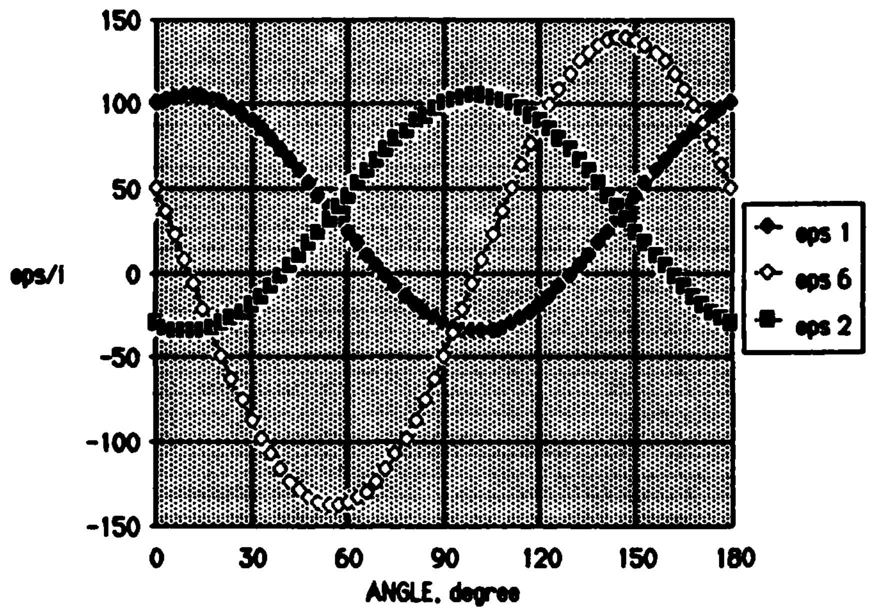


FIGURE 5.6 TRANSFORMED STRAIN COMPONENTS OF EQUATION 5.22.

5.5 CONCLUSIONS – The stress and strain transformations are simple algebraic equations. We want to emphsize the definition of the angle: θ is positive if the new axis is reached by a counter-clockwise rotation. This is shown in Figure 5.1. An inverse transformation, shown in Equation 5.3, for example, merely changes the sign of angle θ, therefore $\sin\theta$.

The concept of principal stress is sometimes used in the failure criteria of isotropic materials. For anisotropic materials, principal stress or strain do not offer anything special in failure criteria. But for highly directional materials, principal stress can be effectively used as an efficient orientation to carry certain combined stresses. This will be discussed further when we talk about design. The sign of the angle is critical: we must know if the laminate is to swing forward or backward.

Section 6

PLY STIFFNESS

6.1 TRANSFORMATION OF STIFFNESS – The on-axis plane stress stiffness and compliance of a unidirectional or fabric ply can be computed from engineering constants:

$$[Q] = \begin{vmatrix} E_x/(1-\nu_x\nu_y) & \nu_x Q_{yy} & 0 \\ \nu_y Q_{xx} & E_y/(1-\nu_x\nu_y) & 0 \\ 0 & 0 & E_s \end{vmatrix} \qquad (6.1)$$

where $Q_{12} = Q_{21}$, $Q_{16} = Q_{61} = 0$, $Q_{26} = Q_{62} = 0$ (6.2)

$$[S] = \begin{vmatrix} 1/E_x & -\nu_y/E_y & 0 \\ -\nu_x/E_x & 1/E_y & 0 \\ 0 & 0 & 1/E_s \end{vmatrix} \qquad (6.3)$$

where $S_{12} = S_{21}$, $S_{16} = S_{61} = 0$, $S_{26} = S_{62} = 0$ (6.4)

The off-axis stiffness matrix can be derived from the stress-strain relation, Equation 4.6, and stress and strain transformation equations 5.1:

$$\{\sigma\} = [Q]\{\epsilon\},\ \ [J]\{\sigma\} = [J][Q]\{\epsilon\},\ \{\sigma'\} = [J][Q][J^T]\{\epsilon'\},$$

therefore $\{\sigma'\} = [Q']\{\epsilon'\}$,

where $[Q'] = [J][Q][J^T]$ (6.5)

This stiffness matrix transformation is to go from the 1-axis to the

1'-axis, with the angle of rotation positive in the counter-clockwise direction. This is shown as angle ϕ in Figure 6.1. The material symmetry axes coincide with the 1-2 axes.

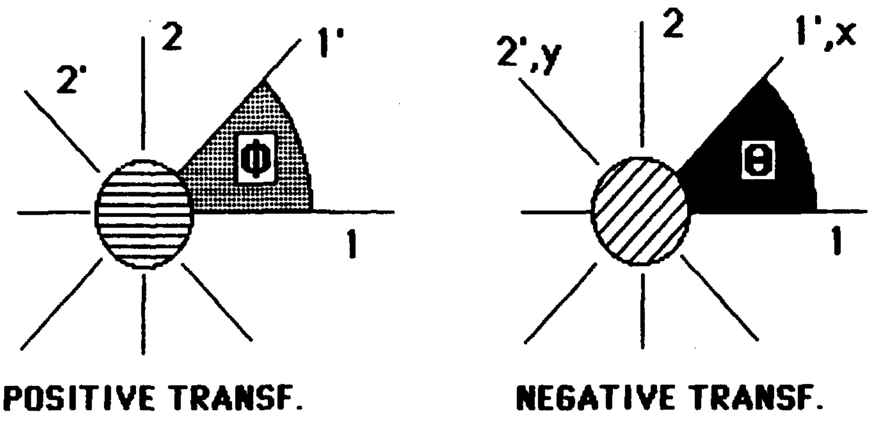


FIGURE 6.1 ANGLES FOR POSITIVE AND NEGATIVE TRANSFORMATIONS. MATERIAL SYMMETRY OR FIBER DIRECTION IS SHOWN IN THE SMALL CIRCLE.

The transformation relation in Equation 6.5 can be shown in the following matrix multiplication table:

TABLE 6.1 POSITIVE TRANSFORMATION OF STIFFNESS MATRIX

On-axis	Q_{xx}	Q_{yy}	Q_{xy}	Q_{ss}
Off-axis				
Q_{11}	m^4	n^4	$2m^2n^2$	$4m^2n^2$
Q_{22}	n^4	m^4	$2m^2n^2$	$4m^2n^2$
Q_{12}	m^2n^2	m^2n^2	m^4+n^4	$-4m^2n^2$
Q_{66}	m^2n^2	m^2n^2	$-2m^2n^2$	$(m^2-n^2)^2$
Q_{16}	$-m^3n$	mn^3	m^3n-mn^3	$2(m^3n-mn^3)$
Q_{26}	$-mn^3$	m^3n	mn^3-m^3n	$2(mn^3-m^3n)$

$m = \cos\phi$, $n = \sin\phi$, ϕ is positive counter-clockwise from the 1- to 1'-axis; see Figure 6.1.

If our materials symmetry axes x-y are off a reference coordinate such as the laminate axes, i.e., the transformation goes from an off-axis ply to a laminate reference coordinate system, negative transformation is required. The angle is then the ply orientation θ. This is labeled negative transformation in Figure 6.1. Any term with odd power of sine must change sign as we go from Table 6.1 to 6.2. This occurs with the 61 and 62 components of the stiffness matrix.

We believe that the negative transformation shown in Table 6.2 is more natural for material properties because it shows the contribution of an off-axis ply to a laminate. Either table can be used but consistency is most important and must be maintained.

TABLE 6.2 NEGATIVE TRANSFORMATION OF STIFFNESS MATRIX

On-axis	Q_{xx}	Q_{yy}	Q_{xy}	Q_{ss}
Off-axis				
Q_{11}	m^4	n^4	$2m^2n^2$	$4m^2n^2$
Q_{22}	n^4	m^4	$2m^2n^2$	$4m^2n^2$
Q_{12}	m^2n^2	m^2n^2	m^4+n^4	$-4m^2n^2$
Q_{66}	m^2n^2	m^2n^2	$-2m^2n^2$	$(m^2-n^2)^2$
Q_{16}	m^3n	$-mn^3$	mn^3-m^3n	$2(mn^3-m^3n)$
Q_{26}	mn^3	$-m^3n$	m^3n-mn^3	$2(m^3n-mn^3)$

$m = \cos\theta$, $n = \sin\theta$, θ is ply orientation, positive in counter-clockwise direction; see Figure 6.1.

The most important relation between the positive and negative transformation or that between Tables 6.1 and 6.2 is:

$$\theta = -\phi \tag{6.6}$$

6.2 MULTIPLE-ANGLE TRANSFORMATION - We can introduce the following linear combinations of the stiffness components so we can express stiffness transformation in multiple angles. The multiple angles are derived from substituting the following trignometric identities into Table 6.2:

$$m^4 = (3+4\cos2\theta+\cos4\theta)/8,\ m^3n = (2\sin2\theta+\sin4\theta)/8$$

$$m^2n^2 = (1-\cos4\theta)/8,\ mn^3 = (2\sin2\theta-\sin4\theta)/8 \qquad (6.7)$$

$$n^4 = (3-4\cos2\theta+\cos4\theta)/8$$

TABLE 6.3 LINEAR COMBINATIONS OF ON-AXIS STIFFNESS MODULI

On-axis Linear comb	Q_{xx}	Q_{yy}	Q_{xy}	Q_{ss}
$U_1 = U_4+2U_5$*	3/8	3/8	1/4	1/2
U_2	1/2	-1/2	0	0
U_3	1/8	1/8	-1/4	-1/2
$U_4 = U_1-2U_5$*	1/8	1/8	3/4	-1/2
$U_5 = (U_1-U_4)/2$*	1/8	1/8	-1/4	1/2

*These combinations are invariant.

The transformation in Table 6.2 can now be rewritten:

TABLE 6.4 TRANSFORMATION OF STIFFNESS IN MULTIPLE ANGLES

Trig func Off-axis	Invariant	$\cos2\theta$	$\cos4\theta$	$\sin2\theta$	$\sin4\theta$
Q_{11}	U_1	U_2	U_3	0	0
Q_{22}	U_1	$-U_2$	U_3	0	0
$Q_{12} = Q_{21}$	U_4	0	$-U_3$	0	0
Q_{66}	U_5	0	$-U_3$	0	0
$Q_{16} = Q_{61}$	0	0	0	$U_2/2$	U_3
$Q_{26} = Q_{62}$	0	0	0	$U_2/2$	$-U_3$

6.3 QUASI-ISOTROPIC CONSTANTS - Associated with every anisotropic material are quasi-isotropic constants derived from the invariants of coordinate transformation. The quasi-isotropic constants represent the lower bound performance of each composite. They are used as a guide in design to insure that whatever ply angle orientations we may select for a given load the laminate performance is at least equal if not better than the quasi-isotropic laminate.

$$[Q]^{iso} = \begin{vmatrix} U_1 & U_4 & 0 \\ U_4 & U_1 & 0 \\ 0 & 0 & U_5 \end{vmatrix} \tag{6.8}$$

$$[S]^{iso} = \begin{vmatrix} U_1/D & -U_4/D & 0 \\ -U_4/D & U_1/D & 0 \\ 0 & 0 & 1/U_5 \end{vmatrix} \tag{6.9}$$

$$\begin{aligned} D &= U_1^2-U_4^2 \\ E^{iso} &= D/U_1 = (U_1^2-U_4^2)/U_1 = U_1[1-(\nu^{iso})^2] \\ \nu^{iso} &= U_4/U_1 \\ G^{iso} &= U_5 \end{aligned} \tag{6.10}$$

6.4 MATRIX INVERSION - $[S] = [Q]^{-1}$

$$\begin{aligned} |Q| &= (Q_{11}Q_{22}-Q_{12}^2)Q_{66}+2Q_{12}Q_{26}Q_{16}-Q_{11}Q_{26}^2-Q_{22}Q_{16}^2 \\ S_{11} &= (Q_{22}Q_{66}-Q_{26}^2)/|Q|,\ S_{22} = (Q_{11}Q_{66}-Q_{16}^2)/|Q|, \\ S_{12} &= (-Q_{12}Q_{66}+Q_{16}Q_{26})/|Q|,\ S_{66} = (Q_{11}Q_{22}-Q_{12}^2)/|Q|, \\ S_{16} &= (Q_{12}Q_{26}-Q_{22}Q_{16})/|Q|,\ S_{26} = (Q_{12}Q_{16}-Q_{11}Q_{26})/|Q|. \end{aligned} \tag{6.11}$$

6.5 ENGINEERING AND COUPLING CONSTANTS -

$$E_1 = 1/S_{11},\ E_2 = 1/S_{22},\ E_6 = G_1 = G_{12} = 1/S_{66},$$

$$\nu_{21} = -S_{21}/S_{11},\ \nu_{12} = -S_{12}/S_{22},$$

$$\nu_{61} = S_{61}/S_{11},\ \nu_{16} = S_{16}/S_{66} \tag{6.12}$$

$$\nu_{62} = S_{62}/S_{22},\ \nu_{26} = S_{26}/S_{66}$$

$$\nu_{21}/\nu_{12} = S_{22}/S_{11} = E_1/E_2$$

$$\nu_{61}/\nu_{16} = S_{66}/S_{11} = E_1/E_6 = E_1/G_1 = E_1/G_{12} \tag{6.13}$$

$$\nu_{62}/\nu_{26} = S_{66}/S_{22} = E_2/E_6 = E_2/G_1 = E_2/G_{12}$$

The coupling constants are derived by using the same normalizing factor for each column of the compliance matrix. As explained in Section 2, we prefer this normalization by columns. But others use the normalization by rows; then they have for example:

$$\nu_{12} = -S_{12}/S_{22},\ \nu_{21} = -S_{21}/S_{11}, \tag{6.14}$$

$$\nu_{12}/\nu_{21} = S_{22}/S_{11} = E_1/E_2$$

Although other authors use this system, we do not do it here.

6.6 TYPICAL PLY DATA

TABLE 6.4 ELASTIC CONSTANTS OF UNIDIRECTIONAL COMPOSITES

Type	CFRP	BFRP	CFRP	GFRP	KFRP
Fiber	T300	B(4)	AS	E-glass	Kev 49
Matrix	N5208	5505	3501	epoxy	epoxy
Eng'g constants					
E_x, GPa	181.00	204.00	138.00	38.60	76.00
E_y, GPa	10.30	18.50	8.96	8.27	5.50
ν_x	0.28	0.23	0.30	0.26	0.34
E_s, GPa	7.17	5.59	7.10	4.14	2.30

Type	CFRP	BFRP	CFRP	GFRP	KFRP
Fiber	T300	B(4)	AS	E-glass	Kev 49
Matrix	N5208	5505	3501	epoxy	epoxy
$[Q]^{(0)}$, GPa					
xx	181.81	204.98	138.81	39.17	76.64
yy	10.35	18.59	9.01	8.39	5.55
xy	2.90	4.28	2.70	2.18	1.89
ss	7.17	5.59	7.10	4.14	2.30
$[S]^{(0)}$, 1/TPa					
xx	5.52	4.90	7.25	25.91	13.16
yy	97.09	54.05	111.61	120.92	181.82
xy	-27.18	-12.43	-33.48	-31.44	-61.82
ss	139.47	178.89	140.85	241.55	434.78
Linear combinations of [Q], GPa					
U_1*	76.37	87.70	59.66	20.45	32.44
U_2	85.73	93.20	64.90	15.39	35.55
U_3	19.71	24.08	14.25	3.33	8.65
U_4*	22.61	28.36	16.96	5.51	10.54
U_5*	26.88	29.67	21.35	7.47	10.95
* invariant					
Quasi-isotropic constants					
E, GPa	69.68	78.53	54.84	18.96	29.02
ν	0.30	0.32	0.28	0.27	0.32
G, GPa	26.88	29.67	21.35	7.47	10.95
$[Q']^{(45)}$, GPa					
11=22	56.66	63.62	45.41	17.12	23.79
12	42.32	52.44	31.21	8.84	19.19
66	46.59	53.76	35.60	10.80	19.60
16=26	42.87	46.60	32.45	7.69	17.77
$\lvert Q'\rvert/1000$	13.43	21.20	8.83	1.34	0.97
$[S']^{(45)}$, 1/TPa					
11=22	59.75	58.90	63.84	93.73	155.20
12	-9.99	-30.55	-6.58	-27.05	-62.19
66	105.71	61.21	123.20	160.30	203.92
16=26	-45.78	-24.58	-52.18	-47.51	-84.33

Type	CFRP	BFRP	CFRP	GFRP	KFRP
Fiber	T300	B(4)	AS	E-glass	Kev 49
Matrix	N5208	5505	3501	epoxy	epoxy
Eng'g constants, θ = 45					
$E_1 = E_2$, GPa	16.74	16.98	15.66	10.67	6.44
E_6, GPa	9.46	16.34	8.12	6.24	4.90
$\nu_{21} = \nu_{12}$	0.17	0.52	0.10	0.29	0.40
$\nu_{61} = \nu_{62}$	-0.77	-0.42	-0.82	-0.51	-0.54
$\nu_{16} = \nu_{26}$	-0.43	-0.40	-0.42	-0.30	-0.41

In the following figures we show the transformation of three stiffness components. From symmetry, the remaining three components can be readily deduced. The last figure is normalized on the specific weight basis with respect to aluminum, steel and titanium. All three metals have the same specific stiffness, and shown as a constant value of unity.

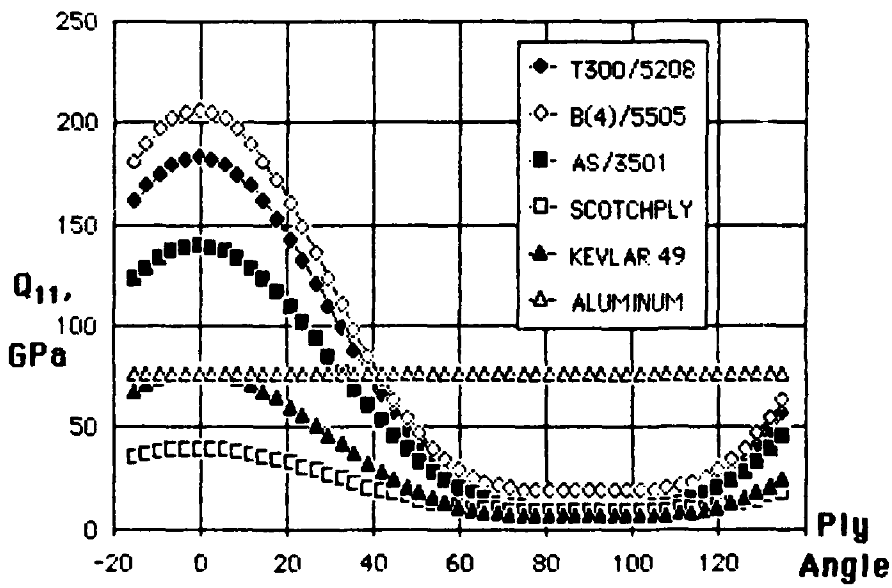


FIGURE 6.2 TRANSFORMATION OF Q_{11} OF VARIOUS COMPOSITES AND ALUMINUM.

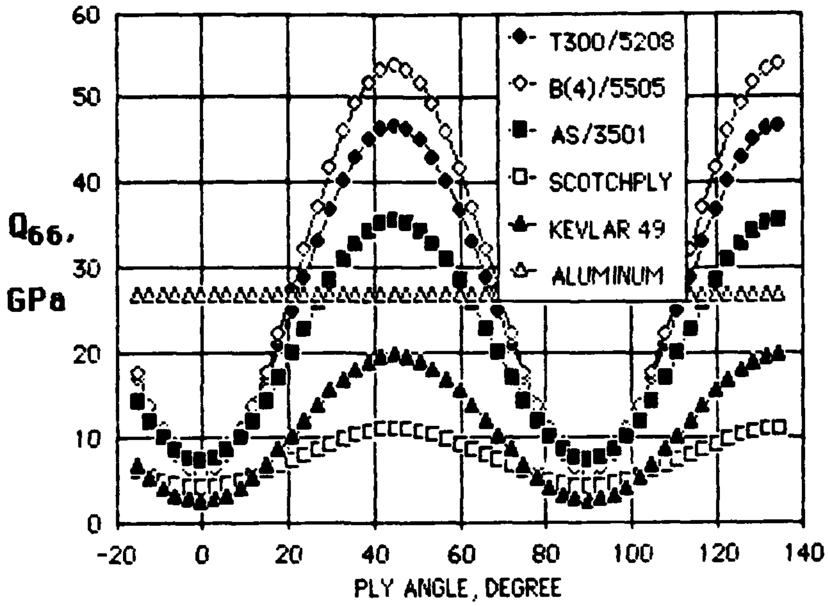


FIGURE 6.3 TRANSFORMATION OF Q_{66} FOR VARIOUS COMPOSITES AND ALUMINUM.

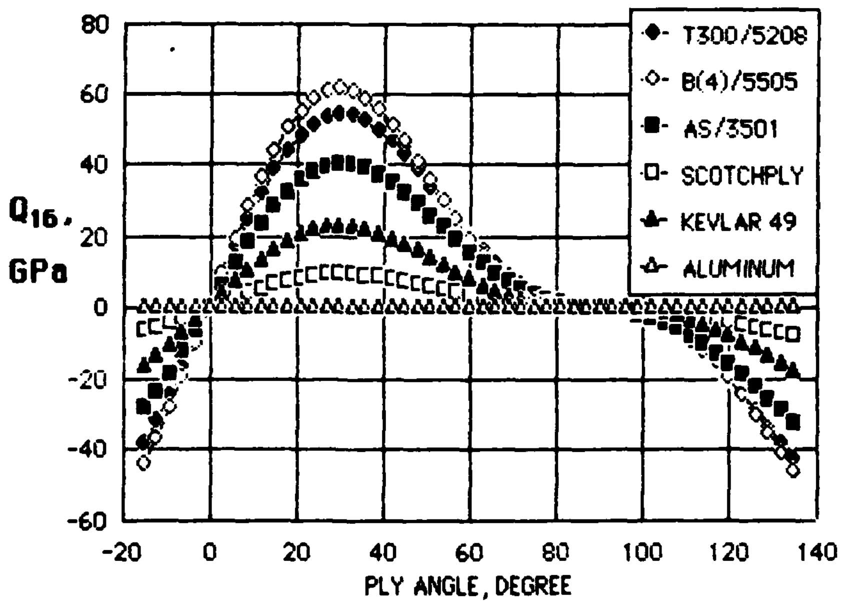


FIGURE 6.4 TRANSFORMATION OF Q_{16} FOR VARIOUS COMPOSITES AND ALUMINUM (WHICH IS ZERO).

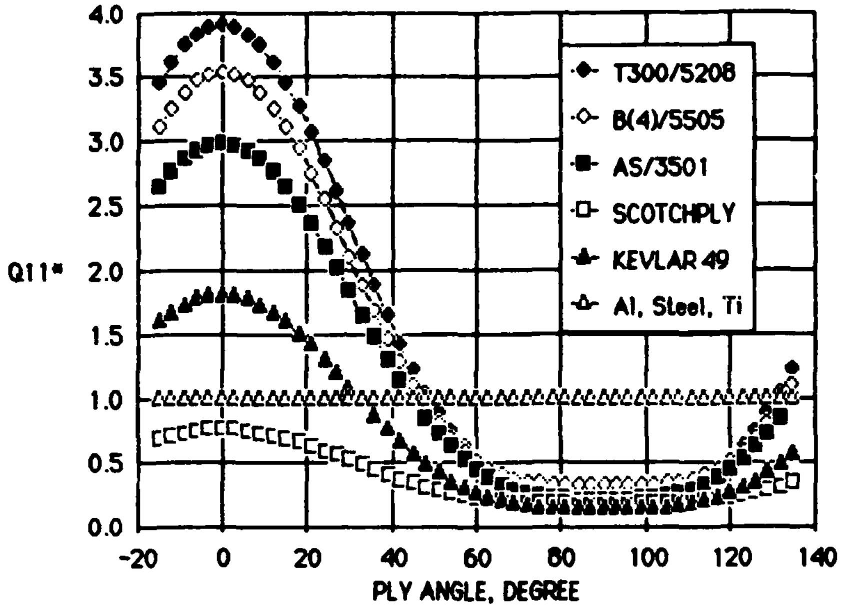


FIGURE 6.5 TRANSFORMATION OF SPECIFIC WEIGHT NORMALIZED Q_{11} AGAINT CORRESPONDING STIFFNESS COMPONENT OF ALUMINUM, STEEL AND TITANIUM.

Section 7

IN-PLANE STIFFNESS OF SYMMETRIC LAMINATES

7.1 LAMINATE CODE – Symmetric laminates have ply angles symmetric about the mid-plane. The conventional laminate code starts the ply angle from the bottom or the top surface of the laminate. For symmetric laminates, the starting surface is of no consequence. But for unsymmetric laminates and for the use of repeating index in the laminate code we must arbitrarily select one surface. We prefer starting from the top surface when repeating index is used.

For the present the following codes represent the same laminate, where subscript T stands for total, and S, symmetric:

$$[0_3/90_2/45/-45_3/-45_3/45/90_2/0_3]_T, \text{ or}$$

$$[0_3/90_2/45/-45_6/45/90_2/0_3]_T, \text{ or}$$

$$[0_3/90_2/45/-45_3]_S \tag{7.1}$$

If a laminate consists of repeating sub-laminates, a number representing the multiple units may be used before the letter T or S. Using the laminate above as a sub-laminate, we can represent a laminate consisting of 4 such units where the repeating index is 2 for this example:

$$[0_3/90_2/45/-45_3]_{2S},\ \{[0_3/90_2/45/-45_3]_2\}_S, \text{ or}$$

$$\{[0_3/90_2/45/-45_3][0_3/90_2/45/-45_3][-45_3/45$$

$$/90_2/0_3][-45_3/45/90_2/0_3]\}_T \tag{7.2}$$

7.2 LAMINATED PLATE THEORY – In addition to the usual assumptions of a linearly elastic material and linear strain-displacement relations, three principal geometric assumptions are:

1. Laminate is symmetric.

2. Laminate is thin:

   $h \ll a, b$ where h is laminate thickness
   a is laminate length
   b is laminate wideth

3. Under in-plane load or deformation, ply strain is constant and equal to the laminate strain:

$$\{\epsilon\} = \{\epsilon^o\}, \text{ or } \epsilon_i = \epsilon_i^o, \ i = 1,2,6 \qquad (7.3)$$

We can define in-plane stress, stress resultant, and laminate stiffness by integration:

$$\{N\} = \int_{-h/2}^{h/2} \{\sigma\}dz = \int_h [Q]\{\epsilon\}dz = \int_h [Q]dz\ \{\epsilon\} \qquad (7.4)$$

$$\{N\} = [A]\{\epsilon\} \qquad (7.5)$$

$$\{\epsilon^o\} = [a]\{N\} \qquad (7.6)$$

$$[A] = \int_{-h/2}^{h/2} [Q]dz \qquad (7.7)$$

$$[a] = [A]^{-1} \qquad (7.8)$$

We define normalized variables:

$$\{\sigma^o\} = \{N\}^* = \{N\}/h,\ [a^*] = [a]h,\ [A^*] = [A]/h \qquad (7.9)$$

The effective stress-strain relations for in-plane behavior are:

$$\{\sigma^o\} = [A^*]\{\epsilon^o\}, \tag{7.10}$$

$$\{\epsilon^o\} = [a^*]\{\sigma^o\} \tag{7.11}$$

The normalized material properties are useful for direct comparison with other materials because the properties are intensive, independent of the thickness of the laminate.

7.3 EFFECTIVE ENGINEERING CONSTANTS

– Engineering constants are defined from the components of normalized compliance:

$$\begin{aligned}
&E_1^o = 1/a_{11}^*,\ E_2^o = 1/a_{22}^*,\ E_6^o = 1/a_{66}^*,\\
&\nu_{21}^o = -a_{21}/a_{11},\ \nu_{12}^o = -a_{12}/a_{22},\\
&\nu_{61}^o = a_{61}/a_{11},\ \nu_{16}^o = a_{16}/a_{66},\\
&\nu_{62}^o = a_{26}/a_{22},\ \nu_{26}^o = a_{26}/a_{66}.
\end{aligned} \tag{7.12}$$

Again, the normalization by columns is used for the coupling constants.

7.4 PLY STRESS AND STRAIN

– For symmetric laminate under in-plane stress or deformation, ply strain and laminate strain are assumed to be equal. Ply stress varies from ply angle to ply angle; see Figure 7.1.

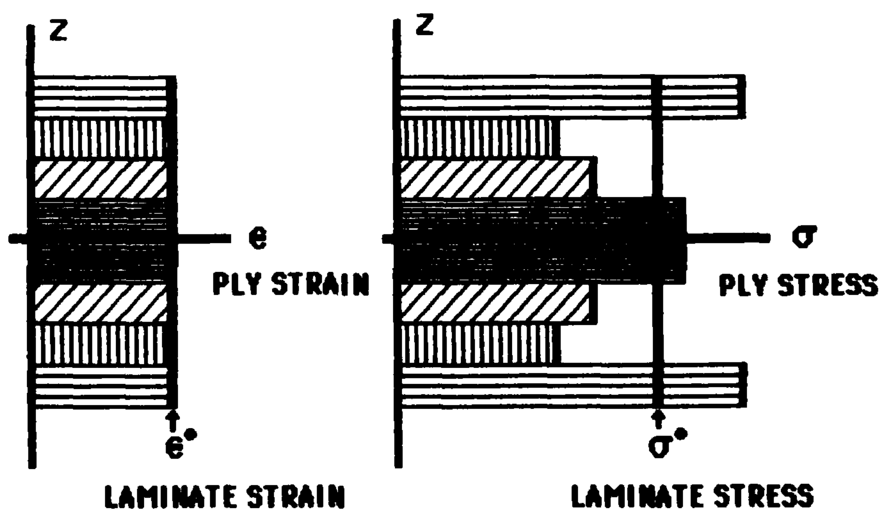


FIGURE 7.1 STRESS AND STRAIN ACROSS LAMINATE THICKNESS.

$\{\epsilon^o\} = \{\epsilon'\}$ = laminate- and off-axis ply strain

$\{\epsilon\}^{(i)}$ = on-axis strain of the i-th ply = $[J^T]^{(i)}\{\epsilon'\}$,

$\{\sigma\}^{(i)}$ = on-axis stress of the i-th ply = $[Q]\{\epsilon\}^{(i)}$,

(7.13)

$[Q]$ = on-axis ply stiffness matrix (Equation 6.1),

$\{\sigma'\} = [Q']\{\epsilon^o\}$ = laminate- or off-axis ply stress,

$[Q']$ = off-axis ply stiffness matrix (Table 6.2).

where from Equation 5.4:

$$[J^T] = \begin{vmatrix} m^2 & n^2 & -mn \\ n^2 & m^2 & mn \\ 2mn & -2mn & m^2-n^2 \end{vmatrix} \tag{7.14}$$

$m = \cos\theta$, $n = \sin\theta$, θ = ply angle, shown in Figure 7.2.

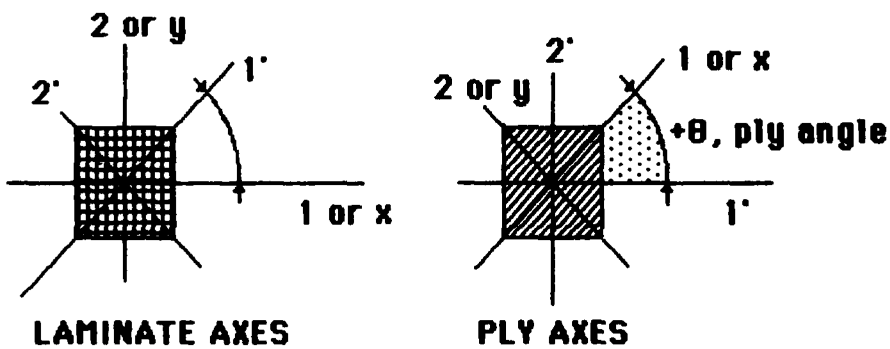


FIGURE 7.2 LAMINATE AND PLY COORDINATE AXES.

The integrated ply stress must be in static equilibrium with the applied loads. This was shown graphically in Figure 7.1.

$$\{\sigma^o\} = 1/h \int_{-h/2}^{h/2} \{\sigma\}dz, \tag{7.15}$$

$$\{\sigma^f\} = 6/h^2 \int_{-h/2}^{h/2} \{\sigma\} z dz = 0 \qquad (7.16)$$

7.5 STIFFNESS MATRIX EVALUATION

– The integration of Equation 7.7 can be performed using many different methods. Three methods will be shown and their features described. We define a ply group as plies with same angle grouped or banded together. For a laminate with m ply groups, the integration can be replaced by a summation:

$$[A] = \sum_{i=1}^{m} [Q']^{(i)}\left[z^{(i)} - z^{(i-1)}\right], \qquad (7.17)$$

$[Q']^{(i)}$ = off-axis stiffness of the i-th ply group with angle θ measured from the laminate axes (Table 6.2 or 6.3).

This is a direct summation replacing the integration. This is a good method if the flexural stiffness and coupling matrices of the laminate are also being evaluated at the same time. From this summation we can develop other methods of integration:

$$[A] = \sum_{i=1}^{m} [Q']^{(i)} h^{(i)}, \qquad (7.18)$$

where $h^{(i)}$ = thickness of the i-th ply group

then $$[A] = \sum_{i=1}^{m} [Q']^{(i)} h_o n^{(i)},$$

where h_o = unit ply thickness,

$n^{(i)}$ = ply number in the i-th group.

Finally $$[A] = \sum_{i=1}^{m} [A^o]^{(i)} n^{(i)}, \tag{7.19}$$

where $[A^o]^{(i)}$ = the i-th unit ply stiffness

$$= [Q']^{(i)} h_o$$

This is the "cash register" method; i.e., only the products of the ply number and the unit ply stiffness, component by component, need to be summed. This is easier to use than the direct summation. We can both add and subtract plies. We can also evaluate hybrids, a laminate with two or more materials. The unit of stiffness is N/m. If we use the laminate code for symmetric laminate, like the last of Equation 7.1, the letter S means doubling the laminate matrix resulting from 7.17. Another method of integration is:

$$[A^*] = [A]/h$$

$$= \sum_{i=1}^{m} [Q']^{(i)} h^{(i)}/h$$

$$[A^*] = \sum_{i=1}^{m} [Q']^{(i)} v^{(i)}, \tag{7.20}$$

where $v^{(i)}$ = fraction of the i-th group.

This is the "rule-of-mixtures" method. Since this normalized stiffness is intensive, independent of the absolute laminate thickness, it is used for direct comparison with other materials, and for materials input data to finite element analysis. Unlike to absolute matrix in Equation 7.19, the resulting normalized matrix need not be doubled because it is not dependent on the laminate thickness.

Section 8

FLEXURAL STIFFNESS OF SYMMETRIC LAMINATES

8.1 LAMINATED PLATE THEORY – Three assumptions analogous to the in-plane behavior of a laminate are:

1. Laminate is symmetric. The laminate may have a honeycomb core with a total thickness measured in number of plies. One half of core thickness in number of unit plies will appear in the laminate code; e.g., for a 6-ply thick core:

$$\{[0_3/90_2/45/-45_3]/c_3\}_S \tag{8.1}$$

   The total laminate is 24-ply thick.

2. Laminate is thin:

   h << a, b where h is laminate thickness,
   a is laminate length, and
   b is laminate width.

3. Under flexural load or deformation ply strain is a linear function of the thickness coordinate z:

$$\{\epsilon\} = z\{k\}, \text{ or } \epsilon_i = zk_i \tag{8.2}$$

We can define moment, flexural stress, laminate flexural stiffness, and effective engineering constants from integration through thickness h of the laminate:

$$\{M\} = \int_{-h/2}^{h/2} \{\sigma\} z\,dz = \int_h [Q]\{\varepsilon\} z\,dz = \int_h [Q] z^2 dz \{k\}$$

$$\{M\} = [D]\{k\} \qquad (8.3)$$

$$\{k\} = [d]\{M\} \qquad (8.4)$$

$$[D] = \int_{-h/2}^{h/2} [Q] z^2 dz \qquad (8.5)$$

$$[d] = [D]^{-1} \qquad (8.6)$$

In terms of normalized variables which are useful in comparing different materials:

$\{\sigma^f\} = 6\{M\}/h^2$ = effective stress at outer surfaces, in Pa (8.7)

$\{\varepsilon^f\} = h\{k\}/2$ = actual strain at outer surfaces (8.8)

$[D^*] = 12[D]/h^3 = [D]/J^*$ = normalized flexural stiffness, in Pa (8.9)

$[d^*] = [d]h^3/12 = [d]\,J^*$ = normalized flexural compliance, in 1/Pa (8.10)

$J^* = h^3/12$ = normalized moment of inertia (8.11)

In the case of a beam J^* is equal to J/b, where b = width. The symbol J here should not be confused with the [J] for coordinate transformation defined in Equation 5.1 et al. and used later in sub-section 8.3.

The effective stress-strain relations in flexure are:

$$\{\sigma^f\} = [D^*]\{\varepsilon^f\} \qquad (8.12)$$

$$\{\varepsilon^f\} = [d^*]\{\sigma^f\} \qquad (8.13)$$

8.2 EFFECTIVE ENGINEERING CONSTANTS -

$$E_1^f = 1/d_{11}^*,\ E_2^f = 1/d_{22}^*,\ E_6^f = 1/d_{66}^*,$$

$$\nu_{21}^f = -d_{21}/d_{11},\ \nu_{12}^f = -d_{12}/d_{22}$$

$$\nu_{61}^f = d_{61}/d_{11},\ \nu_{16}^f = d_{16}/d_{66}, \tag{8.14}$$

$$\nu_{62}^f = d_{62}/d_{22},\ \nu_{26} = d_{26}/d_{66}.$$

Again, the normalization by columns is used for the definition of the coupling constants.

8.3 PLY STRESS AND STRAIN - *Analogous to Equation 7.13 for the ply stress and strain for the in-plane loading, we now have for flexure:*

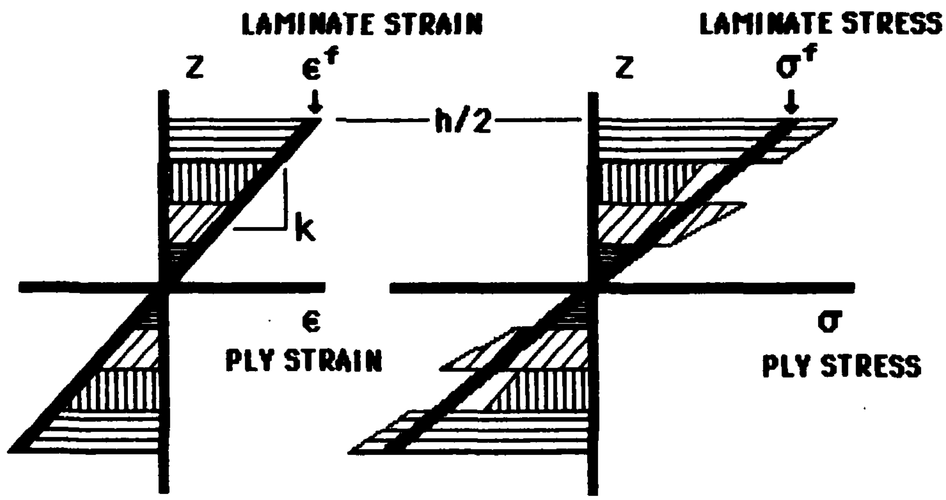


FIGURE 8.1 PLY STRESS AND STRAIN AND THEIR EQUIVALENT LAMINATE STRESS AND STRAIN.

$\{\epsilon'\}^{(z)} = z\{k\}$ = laminate- and off-axis ply strain

$\{\epsilon\}^{(z)} = [J^T]^{(z)}\{\epsilon'\}^{(z)} = [J^T]^{(z)} z\{k\}$

= on-axis ply strain at z for a given ply angle

$\{\sigma\}^{(z)} = [Q]^{(z)}\{\epsilon\}^{(z)}$ = on-axis ply stress at z (8.15)

$[Q]^{(z)}$ = on-axis stiffness matrix at z (Equation 6.1)

$\{\sigma'\}^{(z)} = [Q']^{(z)}\{\epsilon'\}^{(z)}$ = laminate- or off-axis ply stress

$[Q']^{(z)}$ = off-axis stiffness matrix at z (Table 6.2 or 6.3)

where the strain transformation is found in Equation 5.4, repeated here:

$$[J^T] = \begin{vmatrix} m^2 & n^2 & -mn \\ n^2 & m^2 & mn \\ 2mn & -2mn & m^2-n^2 \end{vmatrix} \tag{8.16}$$

The static equilibrium of the ply and laminate stresses must satisfy:

$$\{\sigma^o\} = \{N\}/h = 1/h \int_{-h/2}^{h/2} \{\sigma\}dz = 0 \tag{8.17}$$

$$\{\sigma^f\} = 6\{M\}/h^2 = 6/h^2 \int_{-h/2}^{h/2} \{\sigma\}zdz \tag{8.18}$$

8.4 FLEXURAL STIFFNESS EVALUATION - The integration of Equation 10.5 can be replaced by the summation of a laminate with m ply groups:

$$[D] = 1/3 \sum_{i=1}^{m} [Q']^{(i)}\left[(z^{(i)})^3-(z^{(i-1)})^3\right] \tag{8.19}$$

where $[Q']^{(i)}$ = off-axis stiffness of the i-th ply group with angle θ measured from the laminate axes (Table 6.2 or 6.3). Index i begins from the bottom surface, z=-h/2.

For symmetric laminates:

$$[D] = 2/3 \sum_{j=1}^{m/2} [Q']^{(j)}\left[(z^{(j)})^3-(z^{(j-1)})^3\right] \tag{8.20}$$

The summation may be from either the bottom surface to the mid-plane, or the mid-plane to the top surface of the laminate. We recommend the latter summation for symmetric laminates in which case the ply group must be in reversed order of that in the laminate code. Index j is used here to differentiate from index i used for the entire laminate in Equation 8.19. Unlike the in-plane stiffness, the order of the ply groups is critical for the flexural stiffness. If all plies have the same unit ply thickness and are of the same material, the summation above can be simplified by use of ply number t in place of ply coordinate z:

$$z = th_o$$

where $t = 1,2,3,\ldots n/2$ measured from the midplane,

$$h_o = \text{unit ply thickness} \tag{8.21}$$

Equation 8.17 becomes:

$$[D] = 2h_o^3/3 \sum_{t=1}^{n/2} [Q']^{(t)}\left[t^3-(t-1)^3\right] \tag{8.22}$$

where $$t^3-(t-1)^3 = 3t^2-3t+1 \tag{8.23}$$

We can use the normalized stiffness:

$$[D^*] = [D]/J^* = 12[D]/h^3$$

$$= 12[D]/(nh_o)^3$$

$$[D^*] = 8/n^3 \sum_{t=1}^{n/2} [Q']^{(t)}(3t^2-3t+1) \tag{8.24}$$

where n is the total number of plies; index t, increases with unit plies from the mid-plane. This index is different from index j, which is for the ply groups in Equation 8.18.

8.5 SANDWICH PLATES – For a symmetric laminate with a symmetric core, the ply stress and strain distributions are very similar to those shown in Figure 8.1, except the space provided by the core. This is shown in Figure 8.2.

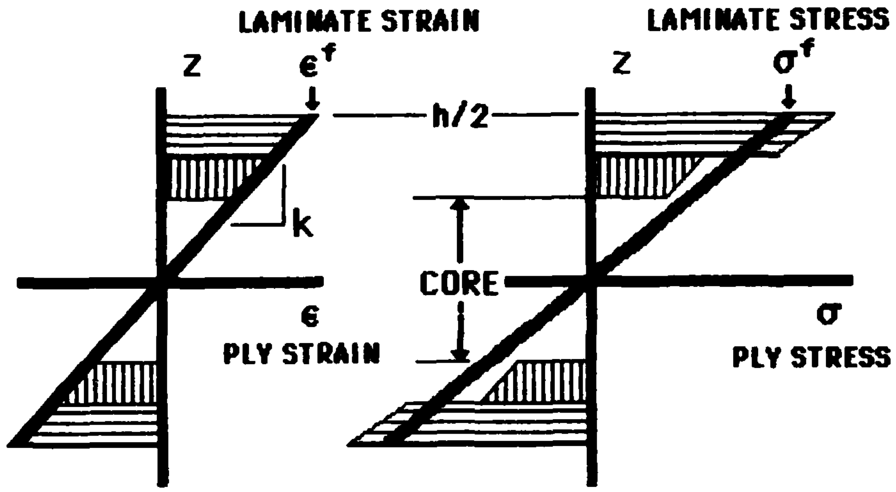


FIGURE 8.2 PLY STRESS AND STRAIN IN A PLATE WITH SANDWICH CORE.

The flexural stiffness is:

$$[D] = 2\int_{z_c}^{h/2} [Q]z^2 dz \tag{8.25}$$

where the top face of the core is z_c. In case of a symmetric laminate with homogeneous plies, the summation can be done for the top half of the laminate, with index j from top face of the core to the top face of the laminate:

$$[D] = 2/3 \sum_{i=c}^{m/2} [Q']^{(j)}\left[(z^{(j)})^3-(z^{(j-1)})^3\right] \tag{8.26}$$

In place of ply groups, a new index t for a ply by ply summation can be done:

$$[D] = 2h_o^3/3 \sum_{t=c}^{n/2} [Q']^{(t)}\left[t^3-(t-1)^3\right] \tag{8.27}$$

If all plies have the same unit ply thickness and are the same material:

$$[D^*] = 8/n^3 \sum_{t=c}^{n/2} [Q']^{(t)}(3t^2-3t+1) \tag{8.28}$$

where c is the half core depth measured in the number of plies; i.e., equal to the subscript after c (for core) in Equation 8.1.

Section 9

STIFFNESS OF UNSYMMETRIC LAMINATES

The laminate code of this most general laminate is typified by letter T for total and ply groups go from the lower surface to the top in accordance with the limits of definite integrals:

$$[0_3/90_2/45]_T \tag{9.1}$$

This convention however is often modified for specialized laminates, such as the symmetric sandwich and the unsymmetric thin wall construction.

9.1 LAMINATED PLATE THEORY – It remains valid if the plate is thin and the strains are linear:

$$\{\epsilon\} = \{\epsilon^o\}+z\{k\}, \text{ or } \epsilon_i = \epsilon_i^o+zk_i,\ i = 1,2,6 \tag{9.2}$$

We define stress resultants and moments as before and derive the generalized stress-strain relation for our unsymmetric laminates:

$$\{N\} = \int_{-h/2}^{h/2} \{\sigma\}dz = \int_h [Q]\{\epsilon\}dz = \int_h [Q]dz\ \{\epsilon^o\}+\int_h [Q]zdz\ \{k\}$$

$$= [A]\{\epsilon^o\}+[B]\{k\} \tag{9.3}$$

$$\{M\} = \int_{-h/2}^{h/2} \{\sigma\}zdz = \int_h [Q]\{\epsilon\}zdz = \int_h [Q]zdz\ \{\epsilon^o\}+\int_h [Q]z^2dz\ \{k\}$$

$$= [B]\{\varepsilon^o\}+[D]\{k\} \tag{9.4}$$

When discrete, homogeneous plies are used, integration can be replaced by summation:

$$[A] = \sum_{i=1}^{m} [Q']\left[z^{(i)}-z^{(i-1)}\right] \tag{9.5}$$

$$[B] = 1/2 \sum_{i=1}^{m} [Q']\left[(z^{(i)})^2-(z^{(i-1)})^2\right] \tag{9.6}$$

$$[D] = 1/3 \sum_{i=1}^{m} [Q']\left[(z^{(i)})^3-(z^{(i-1)})^3\right] \tag{9.7}$$

m = total number of ply groups

[Q'] = off-axis ply stiffness

[B] = coupling matrix between in-plane and flexure.

= 0 when laminate is symmetric

9.2 PARTIAL INVERSION -

$$\begin{aligned} \{\varepsilon^o\} &= [a]\{N\}-[a][B]\{k\} \\ \{M\} &= [B][a]\{N\}+([D]-[B][a][B])\{k\} \end{aligned} \tag{9.8}$$

This partially inverted relation is useful when curvature remains unchanged; i.e.,

$$\{k\} = 0,\ \{\varepsilon^o\} = [a]\{N\} \tag{9.9}$$

for all laminates, symmetric or not. This simple relation is applicable to cylindrical shells subjected to axisymmetric loading such as internal or external pressure, axial tension or compression, and torque. For purpose of saving materials and mass, there is no need to use symmetric wall if the load is axisymmetric.

9.3 COMPLIANCE - The fully inverted stiffness matrix is:

$$\{\epsilon^o\} = [\alpha]\{N\}+[\beta]\{M\}$$
$$\{k\} = [\beta^T]\{N\}+[\delta]\{M\} \tag{9.10}$$

where $[\alpha] = [a]+[a][B]([D]-[B][a][B])^{-1}[B][a]$

$$[\beta] = -[a][B]([D]-[B][a][B])^{-1}$$
$$[\beta^T] = -([D]-[B][a][B])^{-1}[B][a] \tag{9.11}$$
$$[\delta] = ([D]-[B][a][B])^{-1}$$

For symmetric laminates:

$$[B] = 0: [\alpha] = [a], [\delta] = [d] \tag{9.12}$$

9.4 NORMALIZED STRESS-STRAIN RELATIONS - Equations 9.3 and 9.4, and 9.10 can be expressed in normalized quantities:

In-plane stress: $\{\sigma^o\} = \{N^*\} = \{N\}/h$ (9.13)

Flexural stress: $\{\sigma^f\} = \{M^*\} = 6\{M\}/h^2$ (9.14)

Flexural or surface strain: $\{\epsilon^f\} = \{k^*\} = z^{(max)}\{k\} = h\{k\}/2$ (9.15)

Normalized height: $z^* = 2z/h$, $-1 \le z^* \le 1$ (9.16)

Total strain, equal to the actual strain; See Figure 9.1:

$$\{\epsilon\} = \{\epsilon^o\}+z^*\{\epsilon^f\} \tag{9.17}$$

Total average stress (not equal to the actual stress):

$$\{\sigma^*\} = \{\sigma^o\}+z^*\{\sigma^f\} \tag{9.18}$$

Normalized stiffness, all in Pa or psi:

$$[A^*] = [A]/h,\ [B^*] = 2[B]/h^2,\ [D^*] = 12[D]/h^3 \tag{9.19}$$

Normalized compliance, all in 1/Pa or 1/psi:

$$[\alpha^*] = h[\alpha],\ [\beta^*] = h^2[\beta]/2,\ [\delta^*] = h^3[\delta]/12 \tag{9.20}$$

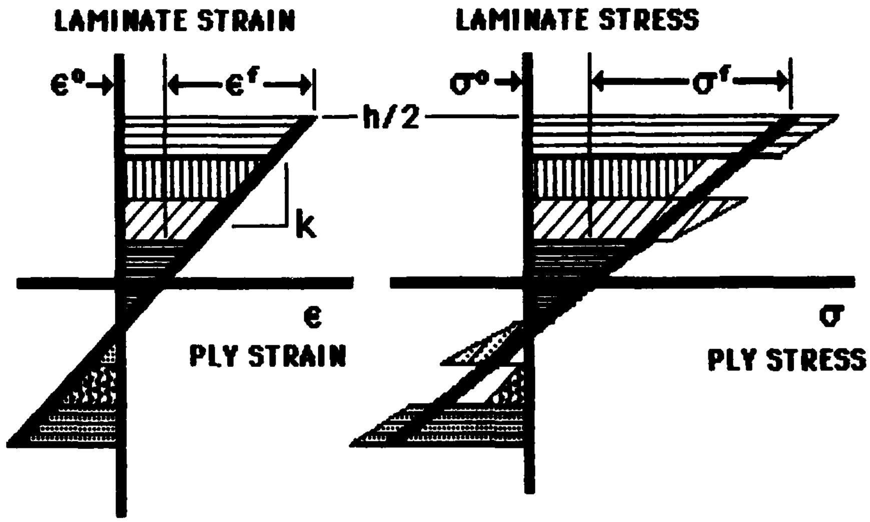


FIGURE 9.1 PLY STRESS AND STRAIN AND LAMINATE STRESS AND STRAIN.

The final normalized stress-strain relations are (factors of 3 and 1/3 in the coupling terms are the results of normalization):

$$\begin{aligned} \{\sigma^o\} &= [A^*]\{\varepsilon^o\}+[B^*]\{\varepsilon^f\} \\ \{\sigma^f\} &= 3[B^*]\{\varepsilon^o\}+[D^*]\{\varepsilon^f\} \end{aligned} \tag{9.21}$$

$$\begin{aligned} \{\varepsilon^o\} &= [\alpha^*]\{\sigma^o\}+[\beta^*]\{\sigma^f\}/3 \\ \{\varepsilon^f\} &= [\beta^{*T}]\{\sigma^o\}+[\delta^*]\{\sigma^f\} \end{aligned} \tag{9.22}$$

9.5 EFFECTIVE ENGINEERING CONSTANTS – Engineering constants for unsymmetric laminates are uncommon and difficult to measure because in-plane and flexural deformations are coupled. Instead of three strain measurements we now need six, three on each surface. We can then compute:

$$\{\epsilon^o\} = (\{\epsilon^+\}+\{\epsilon^-\})/2$$
$$\{\epsilon^f\} = (\{\epsilon^+\}-\{\epsilon^-\})/2 \qquad (9.23)$$

where superscript + and - refer to the strain at the top and bottom surfaces, respectively. Engineering constants are those derived from simple tests such as uniaxial, pure shear, simple bending or pure twisting. For combined stresses, these constants cannot be defined. The constants should not be improperly used; e.g., direct comparison with equivalent constants of symmetric laminates should not be made.

$$E_1^o = \sigma_1^o/\epsilon_1^o = 1/\alpha_{11}^*, \text{ when } \sigma_1^o \neq 0$$
$$E_2^o = \sigma_2^o/\epsilon_2^o = 1/\alpha_{22}^*, \text{ when } \sigma_2^o \neq 0 \qquad (9.24)$$
$$E_6^o = \sigma_6^o/\epsilon_6^o = 1/\alpha_{66}^*, \text{ when } \sigma_6^o \neq 0$$

$$E_1^f = \sigma_1^f/\epsilon_1^f = 1/\delta_{11}^*, \text{ when } \sigma_1^f \neq 0$$
$$E_2^f = \sigma_2^f/\epsilon_2^f = 1/\delta_{22}^*, \text{ when } \sigma_2^f \neq 0 \qquad (9.25)$$
$$E_6^f = \sigma_6^f/\epsilon_6^f = 1/\delta_{66}^*, \text{ when } \sigma_6^f \neq 0$$

Coupling coefficients are derived from the off-diagonal components in the compliance matrix. There are so many (up 15 off-diagonal components resulting in possible 30 coupling constants) in the case of the most general unsymmetric laminates, we will only show the Poisson's ratios:

$$\nu_{21}^o = -\epsilon_2^o/\epsilon_1^o = -\alpha_{21}/\alpha_{11}, \text{ when } \sigma_1^o \neq 0$$
$$\nu_{12}^o = -\epsilon_1^o/\epsilon_2^o = -\alpha_{12}/\alpha_{22}, \text{ when } \sigma_2^o \neq 0$$
$$\nu_{21}^f = -\epsilon_2^f/\epsilon_1^f = -\delta_{21}/\delta_{11}, \text{ when } \sigma_1^f \neq 0 \qquad (9.26)$$
$$\nu_{12}^f = -\epsilon_1^f/\epsilon_2^f = -\delta_{12}/\delta_{22}, \text{ when } \sigma_2^f \neq 0$$

9.6 PARALLEL AXIS THEOREM – To find the stiffness matrix of a laminate with respect to a plane other than the mid-plane:

$$[A'] = [A]$$

$$[B'] = [B]+d[A] \tag{9.27}$$

$$[D'] = [D]+2d[B]+d^2[A]$$

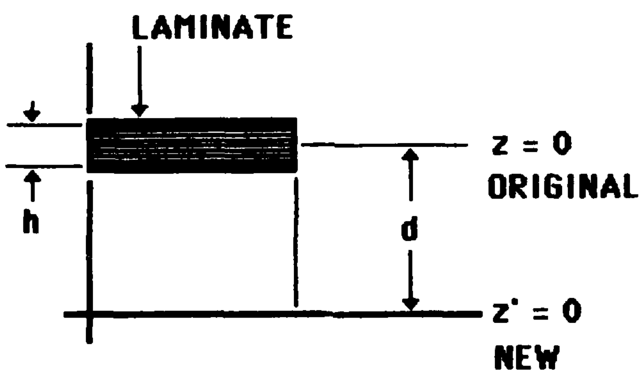


FIGURE 9.2 ORIGINAL AND NEW MID-PLANES FOR PARALLEL AXIS THEOREM.

where primed matrices are the new reference plane; d the distance between the new and old reference planes, see Figure 9.2:

$$d = z'-z = \text{transfer distance} \tag{9.28}$$

This theorem can also be expressed in terms of normalized quantities:

$$d^* = d/h = \text{normalized transfer distance, } h = \text{thickness} \tag{9.29}$$

$$[A^{*\prime}] = [A^*]$$

$$[B^{*\prime}] = [B^*]+2d^*[A^*] \tag{9.30}$$

$$[D^{*\prime}] = [D^*]+12d^*[B^*]+12d^{*2}[A^*]$$

where $[A^*] = [A]/h$, $[B^*] = 2[B]/h^2$, $[D^*] = 12[D]/h^3$, as in Equation 9.19.

9.7 REPEATED SUB-LAMINATES – Sub-laminates are those with small number of plies and can be repeated many times to form a thick laminate. A typical sub-laminate may have up to 8 plies and 4 ply angles. The advantages are:

- Easier selection of optimum ply angles.
- A more damage tolerant laminate resulting from maximum splicing or mixing of plies.
- Simplier layup resulting in lower cost and fewer error in production.

The design process consists of two steps: determine first the optimum ply angles, and secondly, the required number of repeating sub-laminates. The laminate code with sub-laminates in brackets, index r for repeat, and z_c for half-depth of sandwich core is:

$$\{[\text{Sub-laminate}]r/z_c\}_S \tag{9.31}$$

where $[\text{Sub-laminate}] = [\theta_1/\#_1, \theta_2/\#_2, ---]$

For unsymmetric laminates, two repeat indices r^+ and r^- can be used:

$$\{[\text{Sub-laminate}]r^+/c/[\text{Sub-laminate}]r^-\}_T \tag{9.32}$$

where c is now the total core depth in number of plies.

In the case of symmetric laminates in Equation 9.31, we can use the parallel axis theorem to derive the laminate stiffness matrix. The definition of terms are shown in Figure 9.3.

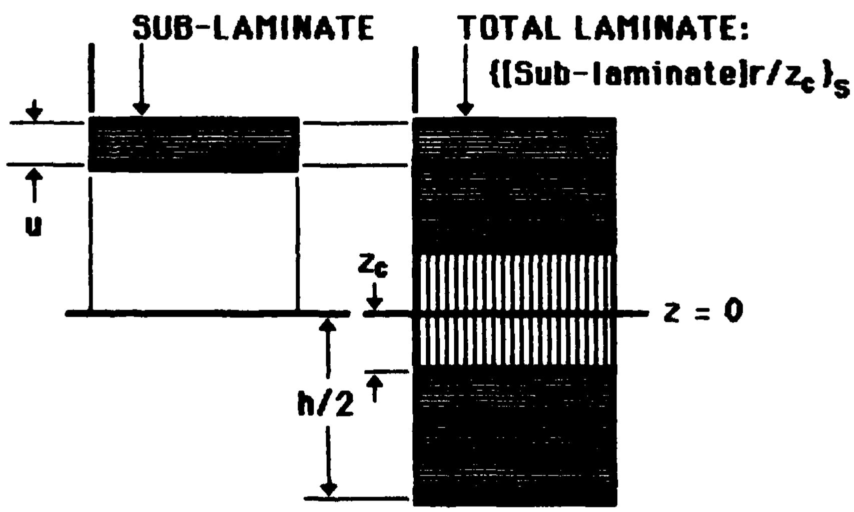


FIGURE 9.3 RELATION BETWEEN SUB-LAMINATE AND TOTAL LAMINATE

The following formulas are derived using the following series for the repeating index r:

$$\Sigma r = r(r+1)/2,\ \Sigma r^2 = r(r+1)(2r+1)/6 \qquad (9.33)$$

The stiffness matrices for the sub-laminate in the location shown in Figure 9.3 (from the top surface) are:

$$[A^o],\ [B^o],[D^o] \qquad (9.34)$$

where Limits of integration are from h/2 to h/2-u.

u = thickness of sub-laminate

The stiffness matrices of the total laminate in Figure 9.3 are:

$$[A] = 2r[A^o]$$

$$[B] = 0 \qquad (9.35)$$

$$[D] = 2r\left[[D^o]-(r-1)u[B^o]+(r-1)(2r-1)u^2[A^o]/6\right]$$

If the total laminate is not symmetric such as that shown in Equation 9.32, the stiffness matrices are:

$$[A] = (r^+ + r^-)[A^o]$$

$$[B] = (r^+ - r^-)\left[[B^o]-(r^+ + r^- - 1)u[A^o]/2\right]$$

$$(9.36)$$

$$[D] = (r^+ + r^-)[D^o]-\left[r^+(r^+ - 1)+r^-(r^- - 1)\right]u[B^o]$$

$$+\left[r^+(r^+ - 1)(2r^+ - 1)+r^-(r^- - 1)(2r^- - 1)\right]u^2[A^o]/6$$

9.8 THIN WALL CONSTRUCTION – A thin wall construction has negligible wall or face sheet thicknesses relative to the total thickness or depth of a sandwich or stiffened construction. The theory of laminated plate theory for symmetric and unsymmetric constructions can be considerably simplified from the complete theory. This simplification makes design of such construction easy to achieve. The typical construction is shown in Figure 9.4:

$$d^+ \approx h/2, \; d^- \approx -h/2 \tag{9.37}$$

Equations 9.27, and 9.30 are simplified because the in-plane stiffness matrix will control the stiffness of the entire construction. The coupling and flexural stiffness matrices make negligible contribution to the total stiffness. But they remain significant in the behavior such as local buckling of the face sheet. The stiffness of the top and bottom sheets are:

$$\begin{aligned} [A,B,D]^+ &= \{1,d^+,(d^+)^2\}[A^+] = \{1,h/2,h^2/4\}[A^+] \\ [A,B,D]^- &= \{1,d^-,(d^-)^2\}[A^-] = \{1,-h/2,h^2/4\}[A^-] \end{aligned} \tag{9.38}$$

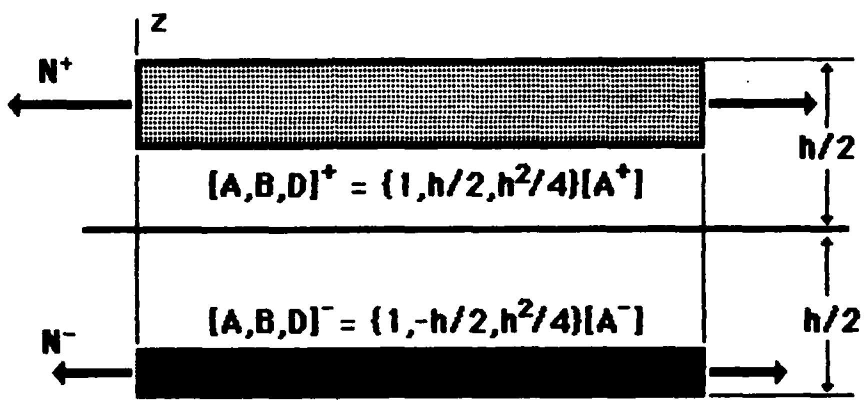


FIGURE 9.4 THIN WALL, UNSYMMETRIC CONSTRUCTION

The total stiffness of this construction is simply the sum of the top and bottom portions in Equation 9.38.

$$\begin{aligned} [A,D] &= \{1,h^2/4\}[A^+ + A^-] \\ [B] &= h/2[A^+ - A^-] \end{aligned} \tag{9.39}$$

When the in-plane stiffness of the face sheets are equal, we have a symmetric construction, for which [B] vanishes.

The in-plane and flexural loads imposed on the construction are:

$$\{N\} = \{N^+ + N^-\}, \; \{M\} = \{N^+ - N^-\}h/2 \tag{9.40}$$

or

$$\{N^+\} = \{N/2 + M/h\}, \; \{N^-\} = \{N/2 - M/h\} \tag{9.41}$$

The resulting stresses and strains at the top and bottom surfaces are:

$$\{N^+\} = [A^+]\{\epsilon^+\},\ \{N^-\} = [A^-]\{\epsilon^-\} \tag{9.42}$$

$$\{\epsilon^+\} = [a^+]\{N^+\},\ \{\epsilon^-\} = [a^-]\{N^-\} \tag{9.43}$$

We can write the in-plane and curvature of our construction as:

$$\{\epsilon^o\} = \{\epsilon^+ + \epsilon^-\}/2,\ \{k\} = \{\epsilon^+ - \epsilon^-\}/h$$

(9.44)

or $$\{\epsilon^+\} = \{\epsilon^o\} + h\{k\}/2,\ \{\epsilon^-\} = \{\epsilon^o\} - h\{k\}/2$$

By combining the relations above, we can show in addition to Equation 9.39 the following stiffness and compliance matrices:

$$[\alpha,\delta] = \{1/4, 1/h^2\}[a^+ + a^-]$$

(9.45)

$$[\beta] = [a^+ - a^-]/2h$$

In normalized terms, we have:

$$[A^*, D^*] = \{1,3\}[A^+ + A^-]/h$$

(9.46)

$$[B^*] = [A^+ - A^-]/h$$

$$[\alpha^*, \delta^*] = \{1, 1/3\}[a^+ + a^-]h/4$$

(9.47)

$$[\beta^*] = [a^+ - a^-]h/4$$

These relations for unsymmetric construction are much readily obtained than the complete laminated plate theory which requires a 6x6 inversion; see Equation 9.11.

9.9 THIN WALL CONSTRUCTION WITH SUB-LAMINATES -

If our thin wall construction consists of the same sub-laminate for the top and bottom faces, with however different repeating indices, say, r^+ and r^-; respectively, we have:

$$[A^+] = r^+[A^o],\ [A^-] = r^-[A^o]$$

(9.48)

$$[a^+] = 1/r^+[a^o],\ [a^-] = 1/r^-[a^o]$$

$$[A^{o*}] = [A^o]/u, \ [a^{o*}] = u[a^o] \tag{9.49}$$

The use of sub-laminates simplifies the calculation of the stiffness of thick laminates. It does not introduce any error. The thin wall construction, on the other hand, is intended only for use of the particular configuration. Errors are introduced if the walls are thick.

The stress-strain relations of this special thin wall construction are:

$$\begin{aligned} [A,D] &= (r^+ + r^-)\{1, h^2/4\}[A^o] \\ [B] &= (r^+ - r^-)[A^o]h/2 \end{aligned} \tag{9.50}$$

$$\begin{aligned} [\alpha,\delta] &= (1/r^+ + 1/r^-)\{1/4, 1/h^2\}[a^o] \\ [\beta] &= (1/r^+ - 1/r^-)[a^o]/2h \end{aligned} \tag{9.51}$$

In normalized terms, we have:

$$\begin{aligned} [A^*,D^*] &= (r^+ + r^-)\{1,3\}[A^{o*}]u/h \\ [B^*] &= (r^+ - r^-)[A^{o*}]u/h \end{aligned} \tag{9.52}$$

$$\begin{aligned} [\alpha^*,\delta^*] &= (1/r^+ + 1/r^-)\{1/4, 1/12\}[a^{o*}] \\ [\beta^*] &= (1/r^+ - 1/r^-)[a^{o*}]/2 \end{aligned} \tag{9.53}$$

Note that when the construction is symmetric; i.e., $r^+ = r^-$, the coupling matrices vanish. The flexural stiffness is three times the in-plane stiffness for thin wall construction, for both symmetric and unsymmetric constructions.

9.10 ACCURACY OF THE THIN WALL THEORY – The simplified theory of thin wall laminate is useful for design but the limits of its utility must be clearly understood. We will make the following comparisons between the unabridged and simplified theories to illustrate the errors introduced by our simplification. We will compute the elastic moduli for a thin wall laminate:

$$[90/c_8/0] \text{ and } [90/c_{98}/0] \tag{9.54}$$

These laminates have 80 and 98 percent core, respectively. The results using the unabridged theory of Equations 9.5-7 and 9.11, and the normalization by Equation 9.19, and the results of the thin wall theory of Equations 9.46 and 47 are listed as follows:

	$[90/c_8/0]$, $z_c^* = .8$			$[90/c_{98}/0]$, $z_c^* = .98$		
	UNA-BRIDGED	THIN WALL	PERCENT ERROR	UNA-BRIDGED	THIN WALL	PERCENT ERROR
A_{11}^*	19.2	19.1	0.0	1.92	1.91	0.0
B_{11}^*	15.4	17.1	11.0	1.70	1.71	0.6
D_{11}^*	47.7	57.3	20.0	5.65	5.73	1.4
α_{11}^*	252	256	0.2	2564	2560	0.1
β_{11}^*	249	229	8.7	220	229	0.4
δ_{11}^*	103	86	21.0	872	855	2.0

Based on the comparison above, we can say that a thin wall construction with 98 percent core will have about 2 percent error. For "thick wall" construction, say, 80 percent core, the error of simplified theory can bring 20 percent error.

Section 10

FAILURE CRITERIA

10.1 INTRODUCTION – Most frequently used criteria are extensions of similar criteria for isotropic materials. These criteria are empirical and are not as analytic as the formulation of the elastic deformation of previous sections. But failure criteria are justified from the standpoint of utility for design and for guidelines for materials improvement. They do not necessarily reflect the failure modes

Failure criteria of an on-axis ply is relatively ease to determine, similar to the elasticity of individual plies. Once the on-axis failure criteria are determined, the off-axis or laminate-axis criteria can be obtained by the coordinate transformation of stress or strain.

The effects of curing stress are additive to the mechanically applied stress following the conventional analysis of thermoelasticity. The temperature and moisture dependent properties can also be readily integrated into our stress analysis if these properties are essentially time-independent. If properties are time-dependent and nonlinear more complex mathematical models than ours will be required.

10.2 BASIC STRENGTH DATA – It is assumed that the following strengths of a unidirectional or fabric ply can be determined from relatively simple tests:

X - Longitudinal tensile strength
X' - Longitudinal compressive strength
Y - Transverse tensile strength (10.1)
Y' - Transverse compressive strength
S - Longitudinal shear strength

Using these data we hope to establish failure criteria that can predict the strength of an orthotropic ply subjected to combined stresses or strains.

10.3 STRENGTH/STRESS RATIO – The strength/stress ratio R or strength ratio, for short, is the ratio between the maximum, ultimate or allowable strength and the applied stress. We postulate that our material is linearly elastic, then for each state of combined stresses there is a state of combined strains. We also apply proportional loading; i.e., all components of stress and strain increase by the same proportion.

$$\{\sigma\}_{max} = R\{\sigma\}_{applied}, \text{ and} \tag{10.2}$$

$$\{\epsilon\}_{max} = R\{\epsilon\}_{applied} \tag{10.3}$$

Numerically R can be any positive value but only that with value greater or equal to unity has physicically meaning. This ratio can be used in a variety of ways to aid design. It is applicable to all common failure criteria.

- Failure occurs when R = 1.
- Safety margin is 1.5 for example when R = 1.5; i.e., the applied stress can increase by a factor of 1.5 before failure occurs.
- Applied stress has exceeded the strength, for example, by a factor of 2 when R = .5. This is not physically possible but is useful information for design.
- When the applied stress or strain component is unity, the resulting strength ratio is the strength. This is an easy method of calculating strength.
- If a laminate has 10 plies and the computed strength ratio is 0.5 for a given in-plane load, the number of plies required to carry the load is simply the number of plies divided by R; i.e., 10/0.5 = 20 plies.

Proportional loading means that the loading vectors in stress and strain space are kept in the same direction as those from the origin. If initial or residual stresses are to be included, the mechanical stress vector will be applied to a point different from the origin. Modification to the strength ratio calculation will have to be made.

We will examine three common failure criteria: maximum stress, maximum strain, and quadratic criteria. In the first two criteria, the strength ratio is applied to each stress or strain component in the symmetry or on-axis orientation of the ply. The lowest ratio of the three components controls. In the quadratic criterion, each combined state of stress and the corresponding state of strain has a unique strength ratio. It can be applied in any reference axes, on or off the ply symmetry axes.

10.4 MAXIMUM STRESS CRITERION – This criterion is applied by calculating the strength/stress ratio for each stress component. The sign of the normal stress determines if tensile or compressive strength should be used. (Primed numbers mean compressive.) The lowest strength ratio among the following three equations determines the ratio that controls the failure of the ply:

$$R_x = X/|\sigma_x| \text{ if } \sigma_x>0, \text{ or } R_x' = X'/|\sigma_x| \text{ if } \sigma_x<0 \qquad (10.4)$$

$$R_y = Y/|\sigma_y| \text{ if } \sigma_y>0, \text{ or } R_y' = Y'/|\sigma_y| \text{ if } \sigma_y<0 \qquad (10.5)$$

$$R_s = S/|\sigma_s| \qquad (10.6)$$

From orthotropic symmetry consideration, shear strength is not sign dependent. It is often assumed that there are five independent modes of failure, one assigned to each positive and negative stress component, such as longitudinal tensile, longitudinal compressive, etc. Such simple identification of failure modes is not reliable because failure processes are highly interacting and complex.

10.5 MAXIMUM STRAIN CRITERION – This is very similar to the maximum stress criterion. The maximum strain from each simple test is either measured or computed from the measured strength divided by the tangent modulus:

$$\epsilon_x^* = X/E_x, \text{ or } \epsilon_x^{*\prime} = X'/E_x$$

$$\epsilon_y^* = Y/E_y, \text{ or } \epsilon_y^{*\prime} = Y'/E_y \qquad (10.7)$$

$$\epsilon_s^* = S/E_s$$

Strength ratio of this criterion is decided from the three ratios of the maximum divided by the applied strain, similar to the maximum stress criterion.

$$R_x = \epsilon_x{}^*/\epsilon_x \text{ if } \epsilon_x>0, \text{ or } R_x' = \epsilon_x{}^{*\prime}/\epsilon_x \text{ if } \epsilon_x<0$$

$$R_y = \epsilon_y{}^*/\epsilon_y \text{ if } \epsilon_y>0, \text{ or } R_y' = \epsilon_y{}^{*\prime}/\epsilon_y \text{ if } \epsilon_y<0 \qquad (10.8)$$

$$R_s = \epsilon_s{}^*/\epsilon_s$$

Like the maximum stress criterion, the sign of the normal strain component determines whether the tensile or compressive ultimate strain should be used. A failure mode is also implicitly assigned to each strain component. Interaction among the possible five modes is not permitted by this and the maximum stress criteria. Because Poisson's ratio is not zero, there is always coupling between the normal components which leads to disagreement between these two criteria as to both the magnitude of the load and the assigned mode responsible for the failure. Agreements between the two criteria exist only on the shear plane and along the four lines of constant failures due to uniaxial stresses. Since deformation of a body is always coupled, by the nonzero Poisson's ratio, we would like to think that failure is also coupled. This is a major shortcoming of the uncoupled failure criteria of maximum stress and maximum strain.

10.6 QUADRATIC CRITERION IN STRESS SPACE

– An easy way to incorporate a coupled or interacting failure criterion is to use the quadratic criterion, which can be rationalized as a generalization of strain or distortional energy. We can no longer assign independent modes of failure because of the coupling. We will further postulate that the criterion in stress space is as follows:

$$F_{ij}\sigma_i\sigma_j + F_i\sigma_i = 1, \; i,j = 1,2,3,4,5,6 \qquad (10.9)$$

For a thin orthotropic ply under plane stress relative to the symmetry axes x-y, the strength parameters F's can be computed from the following:

$$F_{xx} = 1/XX', \; F_{yy} = 1/YY', \; F_{ss} = 1/S^2$$

$$F_x = 1/X - 1/X', \; F_y = 1/Y - 1/Y' \qquad (10.10)$$

$$F_{xy} = F_{xy}{}^*[F_{xx}F_{yy}]^{1/2}, \; -1/2 \le F_{xy}{}^* \le 0$$

In the absence of data from a reliable biaxial test, the normalized interaction term $F_{xy}{}^*$ can be treated as an empirical constant. It is reasonable to have it bounded by the generalized von Mises criterion of -1/2, and zero which yields nearly identical result of Hill's criterion. We will call the latter the modified Hill criterion. Since each combination of stress components in Equation 10.9 reaches its maximum when the right-hand-side reaches unity, we can substitute Equation 10.2 into 10.9:

$$[F_{ij}\sigma_i\sigma_j]R^2+[F_i\sigma_i]R-1=0 \tag{10.11}$$

The stress components here are the applied. For a given material, the F's are specified. For a given state of applied stresses, the σ's are known. We only need to solve the quadratic equation in R in Equation 10.11. The correct answer is the positive square root in the quadratic formula;i.e.,

$$aR^2+bR-1 = 0 \tag{10.12}$$

where $a = F_{ij}\sigma_i\sigma_j,\ b = F_i\sigma_i$

$$R = -(b/2a)+[(b/2a)^2+1/a]^{1/2} \tag{10.13}$$

The conjugate root from negative square root provides the absolute value of the strength ratio if the sign of all the applied stress components is reversed. This is useful for the bending of symmetric plate because the applied stresses change signs between the positive and negative distance from the midplane.

10.7 QUADRATIC CRITERION IN STRAIN SPACE – The plane stress criterion in Equation 10.9 can be represented in strain space by a straightforward substitution of the stress-strain relation:

$$G_{ij}\epsilon_i\epsilon_j+G_i\epsilon_i = 1 \tag{10.14}$$

where $G_{xx} = F_{xx}Q_{xx}{}^2+2F_{xy}Q_{xx}Q_{xy}+F_{yy}Q_{xy}{}^2$

$$G_{yy} = F_{xx}Q_{xy}{}^2+2F_{xy}Q_{xy}Q_{yy}+F_{yy}Q_{yy}{}^2$$

$$G_{xy} = F_{xx}Q_{xx}Q_{xy}+F_{xy}(Q_{xx}Q_{yy}+Q_{xy}{}^2)+F_{yy}Q_{xy}Q_{yy} \tag{10.15}$$

$$G_{ss} = F_{ss}Q_{ss}{}^2$$

$$G_x = F_xQ_{xx}+F_yQ_{xy}$$

$$G_y = F_xQ_{xy}+F_yQ_{yy}$$

Since we assume that the strength ratio based on combined stresses is equal to that on combined strains, we can determine the strength ratio using the state of strain:

$$[G_{ij}\epsilon_i\epsilon_j]R^2+[G_i\epsilon_i]R-1 = 0 \tag{10.16}$$

We can apply the same solution of this quadratic equation as before:

$$aR^2+bR-1 = 0 \tag{10.17}$$

where $a = G_{ij}\epsilon_i\epsilon_j$, $b = G_i\epsilon_i$

$$R = -(b/2a)+[(b/2a)^2+1/a]^{1/2} \tag{10.18}$$

The representation of failure envelopes in strain space is preferred because strain is specified in the laminated plate theory; i.e., strain is at most a linear function of the thickness. Failure envelopes in strain space are independent of the ply angles that exist in a laminate. The envelopes remain fixed; can thus be regarded as material properties. Failure envelopes in stress space of the individual plies in a multidirectional laminate are functions for each laminate. They cannot be treated as materials properties.

10.8 FAILURE SURFACE OF OFF-AXIS PLIES – We know how to transform an off-axis stress and strain to an on-axis orientation. Failure criteria are usually applied in this fashion. But we can just as easily transforming the failure stress or strain from an on-axis orientation (a point on the failure surface) to an off-axis orientation equal to the particular ply angle. We then have created a point on the failure surface of an off-axis ply. In fact, the off-axis surface can be generated from the on-axis surface through a rigid-body rotation equal to twice the ply angle, as shown in Equations 5.14 and 5.15. This relation is precisely that in the Mohr's circle space. Thus, once a failure envelope is determined in the symmetry or orthotropic axes of a ply, the off-axis plies can be easily generated.

10.9 SUCCESSIVE PLY FAILURES – Whatever failure criterion is used, it is applied to each ply within a laminate. The ply with the lowest strength ratio will fail first, thus, the first-ply-failure. The failure mode is most probably cracks running parallel to the fibers of a unidirectional ply. The cracks can be within the matrix and at the fiber-matrix interface. As load increases, more cracks are generated. These cracks are spaced at certain distance, about 7 times the thickness of the ply group according to a shear lag analysis.

This ply with cracks will change the internal stress distribution of the laminate. They result in local stress concentrations which may lead to cracks or fracture in neighboring plies. The effective stiffness of the laminate will also reduce, but not by any noticeable degree because the transverse and shear moduli of a unidirectional ply is small initially compared with that of the longitudinal modulus. As long as the ply is still embedded in the laminate it will continue to contribute to the stiffness of the laminate, particularly in the direction of the fiber. A ply with transverse cracks is not the same if it had been completely removed from the laminate.

There is no simple, systematic method for assessing the effect of a failed ply. In its absence, it may be reasonable to assume that the effective stiffness of a multidirectional laminate is retained. With this assumption, the failure of the remaining plies can be estimated. This estimate of the successive ply failures can be made using the laminated plate theory as if local failures do not drastically affect the overall stress distribution. The laminate is completely failed when the last-ply-failure has occurred. This estimate can only be applied to the in-plane loading of symmetric laminates. It can only be applied to laminates under flexure if we have a thin wall construction; see Sections 9 and 13.

To overlook the local failures within a laminate is not a conservative approach. But the estimate of the ultimate laminate failure is useful for design. The recommended design allowable is to use the first-ply-failure for the limit load, then the estimated last-ply-failure as the ultimate with a 50 percent margin over the limit load. If the margin is less than 50 percent, the limit load should be 67 percent of the estimated last-ply-failure.

10.10 TYPICAL STRENGTH DATA – We will show in the following table typical strength data for four of the five composite materials, the elastic moduli of which we have shown in the last section. Boron/epoxy has been deleted to make room for two new composite materials: the graphite/PEEK (an ICI thermal plastic matrix) and the Hercules IM6/epoxy.

Type	CFRP	CFRP	GFRP	KFRP	CFRTP	CFRP
Fiber	T300	AS	E-glass	Kev 49	AS 4	H-IM6
Matrix	N5208	3501	epoxy	epoxy	PEEK APC2	epoxy
Engineering Constants, GPa or dimensionless						
Ex	181.00	138.00	38.60	76.00	134.00	203.00
Ey	10.30	8.96	8.27	5.50	8.90	11.20
nu/x	0.28	0.30	0.26	0.34	0.28	0.32
Es	7.17	7.10	4.14	2.30	5.10	8.40
Ply Stiffness, GPa						
Qxx	181.81	138.81	39.17	76.64	134.70	204.14
Qyy	10.35	9.01	8.39	5.55	8.95	11.26
Qxy	2.90	2.70	2.18	1.89	2.51	3.58
Qss	7.17	7.10	4.14	2.30	5.10	8.40

Strength Data, MPa						
X	1500	1447	1062	1400	2130	3500
X'	1500	1447	610	235	1100	1540
Y	40	52	31	12	80	56
Y'	246	206	118	53	200	150
S	68	93	72	34	160	98

Max Strain, E-03						
eps x	8.29	10.49	27.51	18.42	15.90	17.24
eps x'	8.29	10.49	15.80	3.09	8.21	7.59
eps y	3.88	5.77	3.75	2.18	8.99	5.00
eps y'	23.88	22.99	14.27	9.64	22.47	13.39
eps s	9.48	13.10	17.39	14.78	31.37	11.67

TABLE 10.1A TYPICAL STRENGTH DATA FOR VARIOUS COMPOSITE MATERIALS AT ROOM TEMPERATURE.

Type	CFRP	CFRP	GFRP	KFRP	CFRTP	CFRP
Fiber	T300	AS	E-glass	Kev 49	AS 4	H-IM6
Matrix	N5208	3501	epoxy	epoxy	PEEK	epoxy
					APC2	
Stress Parameters, Fij (Generalized von MISES)						
Fxy*	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
Fxx	4E-19	5E-19	2E-18	3E-18	4E-19	2E-19
Fyy	1E-16	9E-17	3E-16	2E-15	6E-17	1E-16
Fxy	-3E-18	-3E-18	-1E-17	-3E-17	-3E-18	-2E-18
Fss	2E-16	1E-16	2E-16	9E-16	4E-17	1E-16
Fx	0E+00	0E+00	-7E-10	-4E-09	-4E-10	-4E-10
Fy	2E-08	1E-08	2E-08	6E-08	8E-09	1E-08
Strain Parameters, Gij						
Gxx	12004	7376	1914	13454	6394	5822
Gyy	10681	7467	18882	47657	4890	14914
Gxy	-3069	-1746	1712	2069	-1584	-495
Gss	11118	5828	3306	4576	1016	7347
Gx	61	39	25	-150	-40	-34
Gy	217	131	198	351	66	125
Stress Parameters, Fij (Modified HILL)						
Fxy*	0	0	0	0	0	0
Fxx	4E-19	5E-19	2E-18	3E-18	4E-19	2E-19
Fyy	1E-16	9E-17	3E-16	2E-15	6E-17	1E-16
Fxy	0E+00	0E+00	0E+00	0E+00	0E+00	0E+00
Fss	2E-16	1E-16	2E-16	9E-16	4E-17	1E-16
Fx	0E+00	0E+00	-7E-10	-4E-09	-4E-10	-4E-10
Fy	2E-08	1E-08	2E-08	6E-08	8E-09	1E-08
Strain Parameters, Gij						
Gxx	15544	9889	3669	23445	8136	9259
Gyy	10882	7630	19258	48380	5005	15104
Gxy	3280	2467	5137	16885	1545	4938
Gss	11118	5828	3306	4576	1016	7347
Gx	61	39	25	-150	-40	-34
Gy	217	131	198	351	66	125

TABLE 10.1B TYPICAL STRENGTH DATA FOR VARIOUS COMPOSITE MATERIALS AT ROOM TEMPERATURE.

10.11 EXAMPLES OF FAILURE ENVELOPES

– We will show typical quadratic failure envelopes in stress and strain spaces. In stress space, the envelope for nearly all composites is highly elongated due to their high longitudinal strength. It is therefore difficult to plot it in stress space as an ellipse. Bar charts of the polar radius of the failure envelope would be one easier way of showing the strength variation as function of ply angles. In Figure 10.1, this is done for CFRP T300/5208 in the first and fourth quadrants in the principal stress space. The basic strength data are taken from Table 9.4, where the interaction term is -0.5 (the generalized von Mises criterion).

theta	MPa	Bar chart
90.0	40	←Y, transverse tensile
75.0	42	
60.0	46	
45.0	57	
30.0	81	
15.0	159	
0.0	1500	
-15.0	757	↑X, longitudinal
-30.0	444	tensile
-45.0	328	
-60.0	275	
-75.0	251	
-90.0	246	←Y', transverse compressive

FIGURE 10.1 POLAR RADIUS OF T300/5208 QUADRATIC FAILURE ENVELOPE ALONG THE LONGITUDINAL TENSILE STRESS.

The failure envelope of the same graphite/epoxy composite is plotted in Figure 10.2 for two shear strain levels at zero and 0.009. In Figure 10.3, the envelope of Scotch-ply is plotted for shear strains at zero and 0.016.

The strain space representation has several advantages:

- It is easier to plot because the envelope is less elongated.
- It is invariant; i.e., remains fixed for all laminates.
- It is dimensionless; the same in SI and English units.

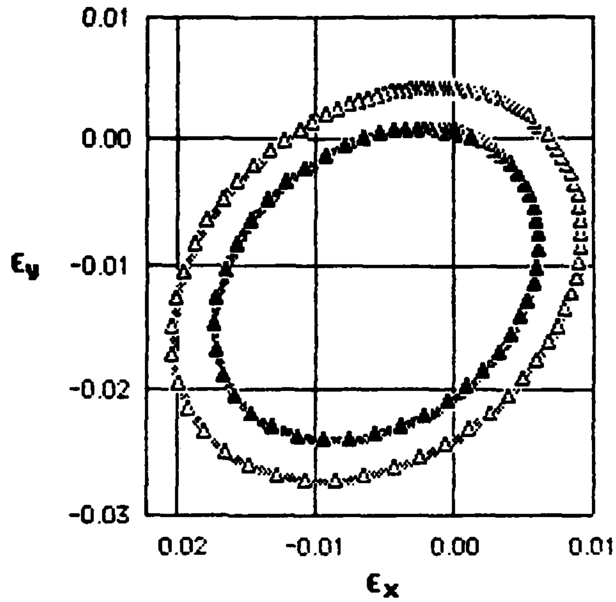


FIGURE 10.2 QUADRATIC FAILURE EVELOPES OF 0 AND 0.009 SHEAR IN STRAIN SPACE OF T300/5208.

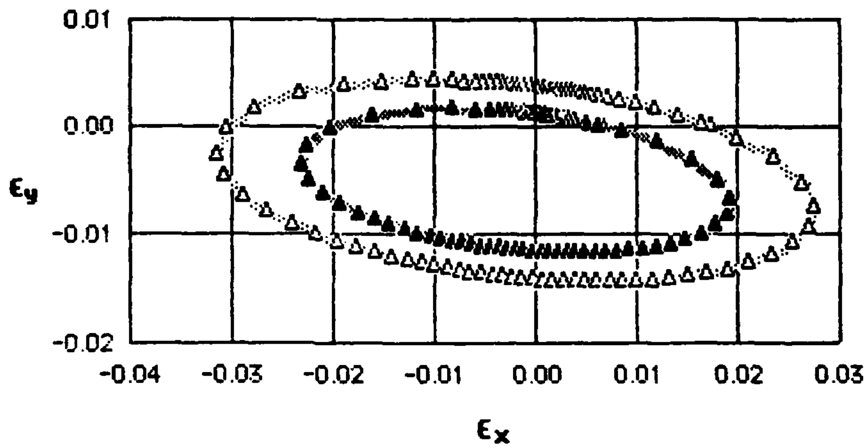


FIGURE 10.3 QUADRATIC FAILURE ENVELOPE OF SCOTCH-PLY FOR SHEAR STRAINS OF ZERO AND 0.0016.

Section 11

MICROMECHANICS

This is a study of the relationship between the physical properties of the constituents and those of the composite. Extensive literature is available. The law or rule of mixtures is the simplest and often sufficiently accurate formula for many micromechanics problems. This rule states that the composite property is the sum of the property of each constituent multiplied by its volume fraction.

All micromechanics analyses are approximate. In addition, many properties of the constituents cannot be readily measured. We therefore believe that the best use of the micromechanics formulas is for sensitivity study; i.e., the change of a known properties due to some micromechanical change. We can be more confident of the relative change than the absolute one.

11.1 DENSITY – The composite density as a function of the constituents and their volume fractions can be computed using the rule of mixtures relation:

$$\rho = v_f \rho_f + v_m \rho_m \qquad (11.1)$$

where ρ = density, v = volume fraction,
Subscript f for fiber, m for matrix.

In terms of mass factions which are often measured and reported:

$$\rho = 1/(m_f/\rho_f + m_m/\rho_m + v_v/\rho)$$

$$v_v = \text{void volume fraction} = 1 - \rho(m_f/\rho_f + m_m/\rho_m) \qquad (11.2)$$

where m = mass fraction of the constituents f and m, or the fiber and the matrix, respectively.

A void content of 0.5 percent is assumed in the Table 11.1. Such low void content has negligible effect on the composite density, and the relation between the volume and mass fractions of each constituent.

TABLE 11.1 VOLUME AND MASS FRACTIONS OF VARIOUS COMPOSITES

Type	CFRP	BFRP	CFRP	GFRP	KFRP	BFRA
Fiber	T300	B(4)	AS	E-glass	Kev 49	Boron
Matrix	N5208	5505	3501	epoxy	epoxy	Al
Fiber density	1.750	2.600	1.750	2.600	1.440	2.600
Matrix dens	1.200	1.200	1.200	1.200	1.200	3.500
ρ_f/ρ_m	1.458	2.167	1.458	2.167	1.200	0.743
void, v_v	0.005	0.005	0.005	0.005	0.005	0.000
Fiber volume	0.700	0.500	0.666	0.450	0.700	0.450
Matrix vol	0.295	0.495	0.329	0.545	0.295	0.550
Comp dens, ρ	1.579	1.894	1.560	1.824	1.362	3.095
Fiber mass	0.776	0.686	0.747	0.641	0.740	0.378
Matrix mass	0.224	0.314	0.253	0.359	0.260	0.622

11.2 LONGITUDINAL YOUNG'S MODULUS AND POISSON'S RATIO

- The micromechanics formula for this stiffness of a unidirectional composite follows the rule-of-mixtures relation:

$$E_x = v_f E_f + v_m E_m \tag{11.3}$$

where E's are the Young's moduli. Since the fiber stiffness is many times the matrix stiffness, the second term in Equation 11.3 can often be ignored:

$$E_x = v_f E_f \tag{11.4}$$

The formula for the longitudinal Poisson's ratio also follows the rule-of-mixtures:

$$\nu_x = v_f \nu_f + v_m \nu_m \tag{11.5}$$

The fiber Poisson's ratio is difficult to determine. We can back calculate from the measured composite Poisson's ratio; i.e.,

$$\nu_f = (\nu_x - v_m \nu_m)/v_f \tag{11.6}$$

The measured Young's modulus and Poisson's ratio of epoxy are:

$$E_m = 3.45 \text{ GPa}, \ \nu_m = 0.35 \tag{11.7}$$

We can now back calculate the fiber Poisson's ratio of the following epoxy-matrix composites:

TABLE 11.2 IMPLIED FIBER POISSON'S RATIOS OF VARIOUS COMPOSITES

Type Fiber Matrix	CFRP T300 N5208	BFRP B(4) 5505	CFRP AS 3501	GFRP E-glass epoxy	KFRP Kev 49 epoxy
ν_x	0.28	0.23	0.30	0.26	0.34
v_f	0.70	0.50	0.66	0.45	0.60
ν_f	0.25	0.11	0.27	0.15	0.33

The implied fiber Poisson's ratios, back-calculated using Equation 11.6 and the measured composite Poisson's ratios listed in Table 6.4, vary from .11 to .33. It is commonly accepted to assume a constant Poisson's ratio for unidirectional composites. It is not a sensitive micromechanical variable.

Woven fabrics have very low Poisson's ratios because the fibers in the transverse direction withhold the Poisson contraction.

11.3 LONGITUDINAL SHEAR MODULUS

- The micromechanics formula for this stiffness modulus also follows the rule-of-mixtures relation except the variables are now the compliance, the reciprocal of the stiffness. One most often cited formula is:

$$1/E_s = v_f/G_f + v_m/G_m \tag{11.8}$$

where E_s is the shear modulus of a unidirectional ply, G's are the shear moduli of the fiber and matrix. This equation however gives value lower than measured data. One way of correcting this deficiency without giving up the simplicity of the rule-of-mixtures relation is to add an empirical constant to this equation. An example of this constant is the easy-to-use stress partitioning parameter proposed by Hahn, where

$$\eta = \text{stress partitioning parameter}$$

$$= (\text{Average matrix stress})/(\text{Average fiber stress})$$

$$= \langle\sigma_m\rangle/\langle\sigma_f\rangle \qquad (11.9)$$

The value of this parameter lies between 0 and 1. We can now modify the rule-of-mixtures relation in Equation 11.8:

$$(1+v^*)/E_S = 1/G_f + v^*/G_m \qquad (11.10)$$

where $v^* = \eta v_m/v_f$ = reduced matrix/fiber volume ratio (11.11)

We recover Equation (11.8) from (11.10) when η is 1. Typical values of the reduced volume ratio v^* are shown and tabulated in following figure and table:

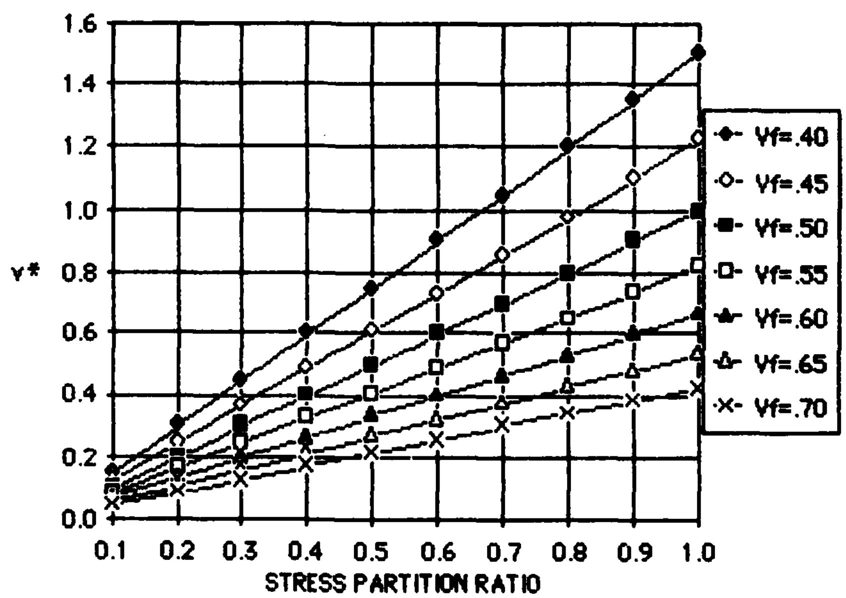


FIGURE 11.1 REDUCED VOLUME RATIO AS A FUNCTION OF STRESS PARTITIONING RATIO OR PARAMETER AND FIBER VOLUMES.

TABLE 11.3 REDUCED VOLUME RATIO AS FUNCTION OF FIBER VOLUME AND STRESS PARTITIONING PARAMETER

η \ V_f	0.40	0.45	0.50	0.55	0.60	0.70	0.75
				v^*			
0.1	0.15	0.12	0.10	0.08	0.07	0.05	0.04
0.2	0.30	0.24	0.20	0.16	0.13	0.11	0.09
0.3	0.45	0.37	0.30	0.25	0.20	0.16	0.13
0.4	0.60	0.49	0.40	0.33	0.27	0.22	0.17
0.5	0.75	0.61	0.50	0.41	0.33	0.27	0.21
0.6	0.90	0.73	0.60	0.49	0.40	0.32	0.26
0.7	1.05	0.86	0.70	0.57	0.47	0.38	0.30
0.8	1.20	0.98	0.80	0.65	0.53	0.43	0.34
0.9	1.35	1.10	0.90	0.74	0.60	0.48	0.39
1.0	1.50	1.22	1.00	0.82	0.67	0.54	0.43

Equation 11.10 can be readily solved graphically if we know the constituent shear moduli and the stress partitioning parameter shown in Figure 11.1 or Table 11.3.

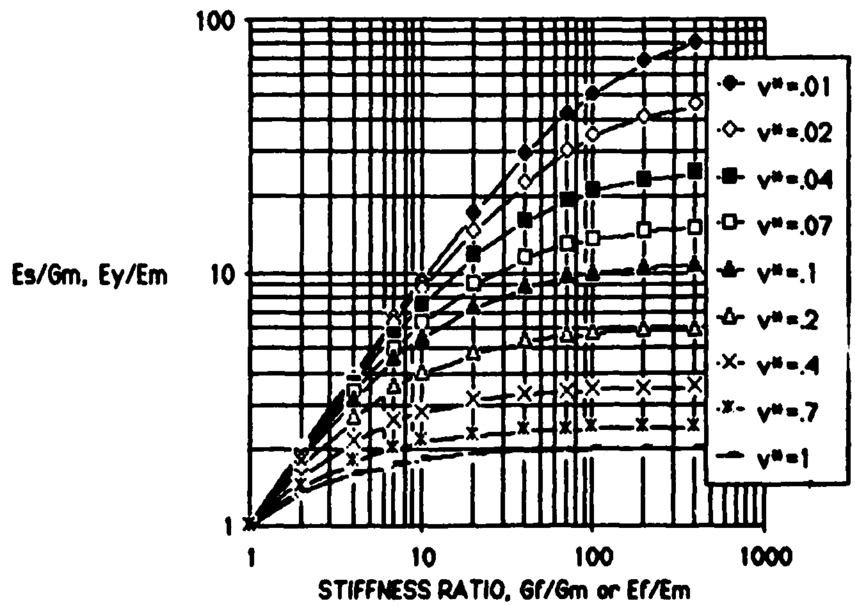


FIGURE 11.2 COMPOSITE SHEAR AND TRANSVERSE MODULI FOR VARIOUS FIBER/MATRIX STIFFNESS AND REDUCED VOLUME RATIOS.

As a guideline of the values for the stress partitioning parameter and the reduced volume ratio, we list in Table 11.4 typical values for several composites. Note that for isotropic fibers such as boron and glass, the assumed stress partitioning parameter is a sensitive variable. It requires high precision in order to have the correct implied fiber modulus.

Fiber	T300	B(4)	AS	Scotch	Kev 49
Matrix	N5208	5505	3501	epoxy	epoxy
Ex, GPa	181.0	204.0	138.0	38.6	76.0
Ey, GPa	10.30	18.50	8.96	8.27	5.50
Nu/x	0.28	0.23	0.30	0.26	0.34
Es, GPa	7.17	5.59	7.10	4.14	2.30
Vf	0.70	0.50	0.66	0.45	0.60
Efx, GPa	258.6	408.0	209.1	85.8	126.7
Em, GPa	3.4	3.4	3.4	3.4	3.4
eta/y	0.500	0.215	0.500	0.516	0.516
v*/y	0.214	0.215	0.258	0.631	0.344
Efy, GPa	18.2	409.8	15.5	85.6	7.0
eta/s	0.400	0.282	0.400	0.316	0.400
v*/s	0.171	0.282	0.206	0.386	0.267
Gfx, GPa	36.7	185.3	160.5	35.5	3.0

TABLE 11.4 BEST FIT STRESS PARTITIONING PARAMETERS FOR TYPICAL COMPOSITES.

11.4 SAMPLE PROBLEMS ON SHEAR MODULUS

– We will give three problem on the computation of the shear modulus of unidirectional composites.

Prob 1: ***Find shear modulus of unidirectional composite assuming isotropic fiber and matrix.***

We assume that our composite is glass and epoxy:

$E_f = 85$ GPa, $\nu_f = 0.2$

$$E_m = 3.4 \text{ GPa}, \nu_m = .35 \tag{11.12}$$

$v_f = .45$

SOLUTION:

From Equation 3.6:

$G_f = 85/2(1+.2) = 35.4$ GPa

(11.13)

$G_m = 3.4/2(1+.35) = 1.26$ GPa

Let $\eta = 0.4$, from Table 11.3: $v^* = 0.49$ for $v_f = .45$

From Equation 11.10:

$E_s = (1+0.49)/(1/35.4+0.49/1.26) = 3.62$ GPa (11.14)

Prob 2: The reported shear modulus of glass/epoxy composite (Scotchply) is 4.14 GPa. This is higher than our calculated value of 3.62 GPa. How can the stress partitioning parameter be adjusted so that the prediction will match the reported value?

SOLUTION:

We rewrite Equation 11.10 to back-calculate the v^*:

$v^* = (1/E_s-1/G_f)/(1/G_m-1/E_s)$ (11.15)

Using the values in Equation 11.12 and reported E_s (=4.14), we can back-calculate v^*:

$v^* = (1/4.14-1/35.4)/(1/1.26-1/4.14) = 0.386$ (11.16)

From Equation 11.11, $\eta = v^* v_f/v_m$ (11.17)

From data in Equation 11.12, $\eta = 0.386(.45/.55) = 0.316$ (11.18)

By substituting the adjusted value of v^* (=0.386), we can recalculate the composite shear modulus using Equation 11.10. The calculated value now is 4.14 GPa, as expected.
The stress partitioning parameter can be seen as an empirical

constant to adjust micromechanics prediction close to reported data. With the new stress partitioning parameter and the reduced volume ratio v*, the micromechanics formula of Equation 11.10 is reliable for performing sensitivity study of micromechanics variables such as constituent properties and volume fractions. This formula and other similar ones are not recommended for predicting absolute composite properties.

Prob 3: Find the shear modulus of a unidirectional T300/5208 composite consisting of anisotropic graphite fiber and isotropic epoxy matrix.

SOLUTION:

The elastic moduli of an anisotropic fiber are not easily measured. Assuming the fiber to be transversely isotropic, there are five independent constants. Only its longitudinal Young's modulus is measured. Its longitudinal shear and transverse moduli are needed for micromechanics predictions but are nearly impossible to measure. *The remaining two* constants can be defined by two Poisson's ratios, which do not have significant contribution to the composite moduli.

It is therefore not practical trying to predict the shear modulus of a composite with anisotropic fibers. The micromechanics formula such as Equation 11.10 is useful for sensitivity studies. We would first establish the missing fiber moduli through back-calculation by assuming reasonable value for the stress partitioning parameter.

The shear and transverse moduli of the fiber can be back-calculated using Equation 11.10, which can be easily rearranged as follows:

$$G_{fx} = \left[(1+v^*)/E_s - v^*/G_m\right]^{-1} \qquad (11.19)$$

The reported longitudinal shear modulus for T300/5208 composite is 7.17 GPa; the shear modulus of epoxy matrix is 1.28 GPa, shown in Equation 11.13; a stress partitioning parameter of 0.4 will be assumed:

parameter of 0.4 will be assumed:

With $v_f = 0.7$, $v^* = 0.4(0.3/0.7) = 0.17$ (11.20)

$G_f = 1/[(1+0.17)/7.17-0.17/1.26] = 36.7$ GPa (11.21)

If we have assumed that the fiber is isotropic, the shear modulus can be calculated using the following moduli:

Longitudinal modulus = 181 GPa

Poisson's ratio = 0.2

From Equation 11.4,

Fiber longitudinal modulus = 181/0.7 = 258 GPa (11.22)

From Equation 3.6,

$G_{f(iso)} = E/2(1+\nu) = 258/2(1+0.2) = 108$ GPa (11.23)

This shear is nearly three times the implied shear in Equation 11.21.

We would recommend the following equation for sensitivity studies of the longitudinal shear modulus of T300/5208:

$E_s = [1+0.4(v_m/v_f)]/[1/36.7+0.4(v_m/v_f)/G_m]$ (11.24)

In this formula we keep the stress partitioning parameter and the fiber shear modulus constant. We can vary the volume fractions and the matrix shear modulus. Only modest variation on these variables are allowed because the stress partitioning parameter and the back-calculated fiber shear modulus will change if the variables in Equation 11.24 change drastically.

Figure 11.3 can be used to calculate the shear moduli of a unidirectional composite with epoxy matrix. This figure is the same as Figure 11.2 except the values are absolute instead of normalized. The reduced volume ratios are the parameter in the figure. It can be obtained in Figure 11.1 or Table 11.3.

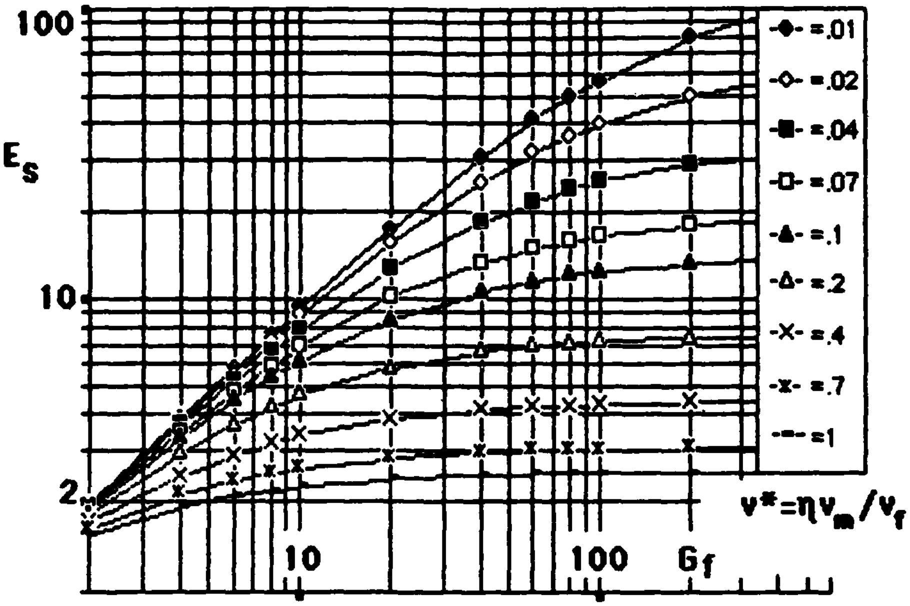


FIGURE 11.5 SHEAR MODULUS OF EPOXY MATRIX UNIDIRECTIONAL COMPOSITES AS FUNCTIONS OF REDUCED VOLUME V*.

11.5 TRANSVERSE MODULUS – The micromechanics method for the transverse modulus follows precisely that for the longitudinal shear modulus. In place of Equations 11.10 et al, we have:

$$(1+v_y^*)/E_y = 1/E_{fy} + v_y^*/E_m \qquad (11.25)$$

where $v_y^* = \eta_y v_m/v_f$ = reduced matrix/fiber volume ratio (11.26)

We have added a subscript y for both the reduced volume ratio and the stress partitioning parameter to differentiate them from those for the longitudinal shear. Their numerical values will be different, as we can see from the best fit data in Table 11.4, where η_y is slighly higher than that for the shear with the exception of the boron/epoxy composite.

The sample problems for the shear modulus in sub-section 11.4 are applicable to the transverse modulus. For example, comparable to Equations 11.15 and 11.19 we now have, respectively:

$$v_y^* = (1/E_{fy} - 1/E_y)/(1/E_y - 1/E_m) \qquad (11.27)$$

$$E_{fy} = \left[(1+v_y^*)/E_y - v_y^*/E_m\right]^{-1} \quad (11.28)$$

Figures 11.1 and 11.3 and Tables 11.3 and 11.4 are applicable for both the transverse and shear moduli. The stress partitioning parameters are different.

Figures 11.4 can be used to calculate the transverse moduli of epoxy matrix unidirectional composites. This figure is the same as Figure 11.2 except the values are absolute instead of normalized.

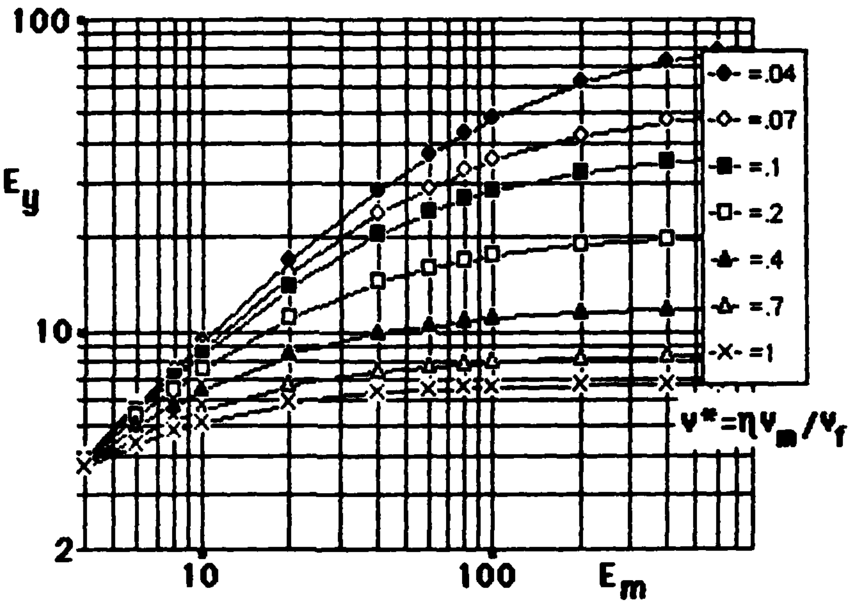


FIGURE 11.6 TRANSVERSE MODULUS OF EPOXY MATRIX UNIDIRECTIONAL COMPOSITES AS A FUNCTION OF REDUCED VOLUME V^*.

11.6 EXPANSION COEFFICIENTS

– The thermal and moisture expansion coefficients of unidirectional epoxy matrix composites can be expressed in the following simplified micromechanics formulas:

$$\alpha_x = \beta_x = 0 \quad (11.29)$$

where the fiber is assumed to be insensitive to temperature and moisture.

$$\alpha_y = v_m(1+\nu_m)\alpha_m, \; \beta_y = v_m(1+\nu_m)\beta_m \quad (11.30)$$

Both transverse thermal and moisture expansions is proportional to v_m.

11.7 STRENGTHS – The micromechanics of strength of unidirectional composites is based on empirical observations. The relations are very approximate and should be even more limited to sensitivity studies than the properties which we have covered in this section earlier.

For the longitudinal tensile strength, it is reasonable to use the rule-of-mixtures relation for sensitivity study; i.e.,

$$X/X_0 = [v_f/v_{f0}]\,[\sigma_f/\sigma_{f0}] \tag{11.31}$$

For the longitudinal compressive strength, we would assume not only its dependence on the longitudinal tensile strength but also the longitudinal shear modulus based on elementary microbuckling of the fibers.

$$X'/X_0' = [v_f/v_{f0}]\,[\sigma_f/\sigma_{f0}]\,[E_s/E_{s0}] \tag{11.32}$$

The transverse tensile and compressive strength and the longitudinal shear strength are assume to by proportional to the matrix strength.

$$Y/Y_0 = Y'/Y_0' = S/S_0 = X_m/X_{mo} \tag{11.33}$$

Section 12

PRINCIPAL STRESS DESIGN

12.1 SINGLE LOADS– When a structure is to be used under one load condition, we can design this structure based on a single load. It is understood that this load results in a combined stress state, with all three components present.

Unidirectional composites have exceedingly high stiffness and strength combined with light weight. But their properties are limited to the direction along the parallel fibers. The load must be uniaxial, not biaxial. When the load is applied slightly off the fiber direction, precipitous drop in properties occurs. We call this material and its properties directional.

Depending on the nature of the single load it is possible to take advantage of the superior properties of a highly directional composite. This design process is easily done by using the principal stress direction. When a structure must carry different loads under different conditions the directional material is in general not adequate. This multiple load condition will require different methods. A method based on the ranking or sorting large number of laminates will be covered in the next section.

12.2 ISOTROPIC DESIGN – The design for isotropic materials is most frequently based on the von Mises failure criterion. For plane stress:

$$\sigma_1^2-\sigma_1\sigma_2+\sigma_2^2+3\sigma_6^2 = X^2 \qquad (12.1)$$

where X *= uniaxial yield or ultimate strength.*

In terms of principal stresses Equation 12.1 becomes:

$$\sigma_I^2-\sigma_I\sigma_{II}+\sigma_{II}^2 = X^2 \qquad (12.2)$$

In Figure 12.1 the yield or failure criterion normalized by the uniaxial strength X is plotted. The tensile and compressive strengths are equal.

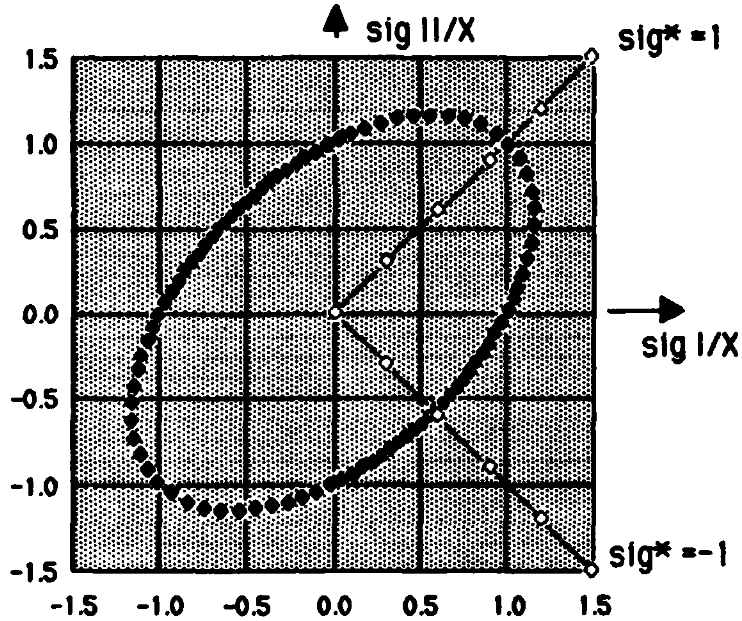


FIGURE 12.1 VON MISES FAILURE CRITERION IN PRINCIPAL PLANE.

Symmetry exists with reference to principal stress ratios of ±1; this is shown as $\sigma^* = \pm 1$ in Figure 12.1. The strength capability as measured by σ_I between these stress ratios is shown in Figure 12.2 for aluminum with tensile strength of 200 MPa.

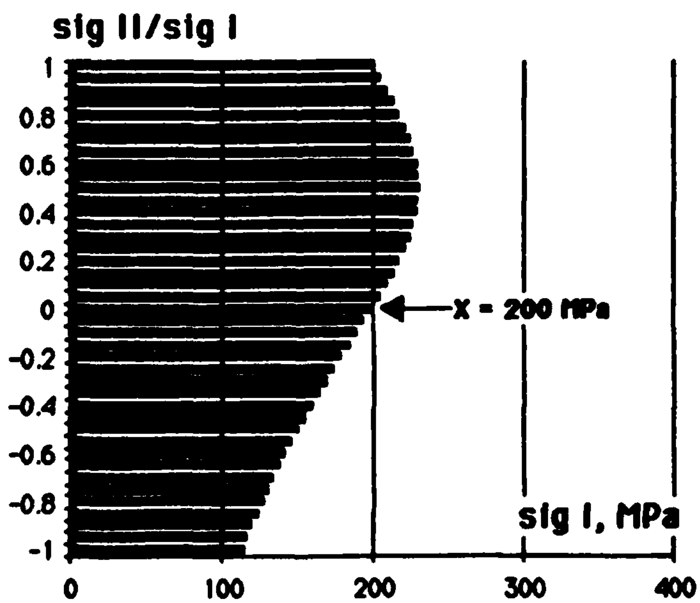


FIGURE 12.2 STRENGTH CAPABILITY UNDER BI-AXIAL STRESSES.

Each strength under biaxial principal stresses in Figure 12.2 corresponds to a point on the failure envelope in Figure 12.1. Like all isotropic materials the strength under biaxial stresses is relatively flat, not highly directional. The maximum and minimum values are at:

$$\begin{aligned} &\text{when stress ratio } \sigma^* = 1/2: \ \sigma_I = (2/\sqrt{3})X = 1.154X \\ &\text{when stress ratio } \sigma^* = -1: \ \sigma_I = (1/\sqrt{3})X = 0.577X \end{aligned} \tag{12.3}$$

The latter corresponds to the shear strength of this material. For our aluminum with a uniaxial strength of 200 MPa:

$$\begin{aligned} &\text{1:2 biaxial strength} = 231 \text{ MPa} \\ &\text{Shear strength} = 115 \text{ MPa} \end{aligned} \tag{12.4}$$

12.3 PRINCIPAL STRESS DESIGN –

For a single load design of composite materials, we recommend the use of principal stress as the criterion. For each state of stress there is a principal direction when shear stress vanishes. This orientation is easily determined, as shown in Section 5.3 and Figure 5.2. The Mohr's circle, repeated here, provides a quick estimate of the principal stresses and the phase angle θ_0:

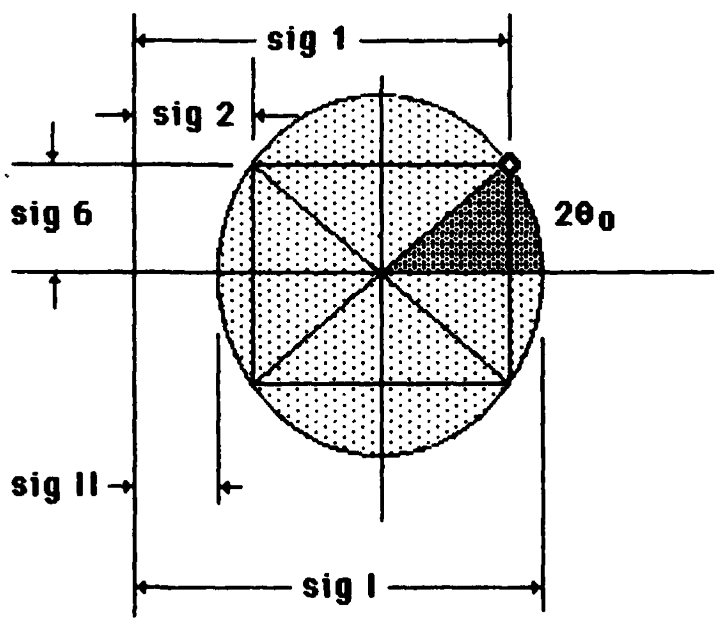


FIGURE 12.3 RELATION BETWEEN APPLIED AND PRINCIPAL STRESSES.

The phase angle can be confusing in two aspects: its sign and the factor of two in the Mohr's circle space. The phase angle in both Figures 12.3 and 12.4 are positive.

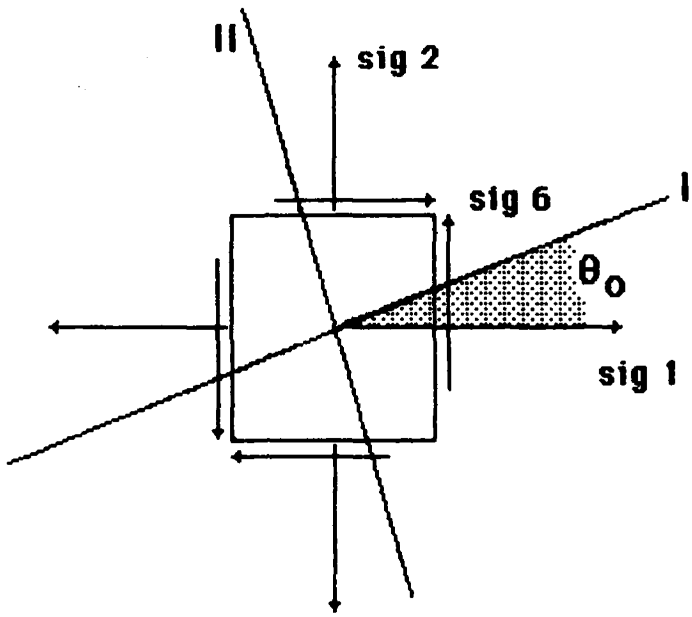


FIGURE 12.4 ORIENTATION OF THE PRINCIPAL AXES FROM THOSE OF THE APPLIED STRESS.

The principal stress design is to utilize the highly directional properties of unidirectional and laminated composites. This concept is useful for single load design.

12.4 DESIGN WITH DIRECTIONAL MATERIALS

– We will show the design options of a CFRP (T300/5208) for single loads. Like the isotropic material, each strength under biaxial principal stresses corresponds to a point on the failure envelope. We use the quadratic failure criterion with a normalized interaction term of -1/2; i.e., the generalized von Mises criterion. The strength data for are taken from Table 9.4. The strengths under combined principal stresses comparable Figure 12.2 are shown in Figure 12.5. The material is highly directional and shows the precipitous change in strength as the stress ratios move away from zero. The drop in strength in the positive stress ratios is greater that that in the negative ratios. This is caused by the transverse tensile strength being smaller than the transverse compressive strength.

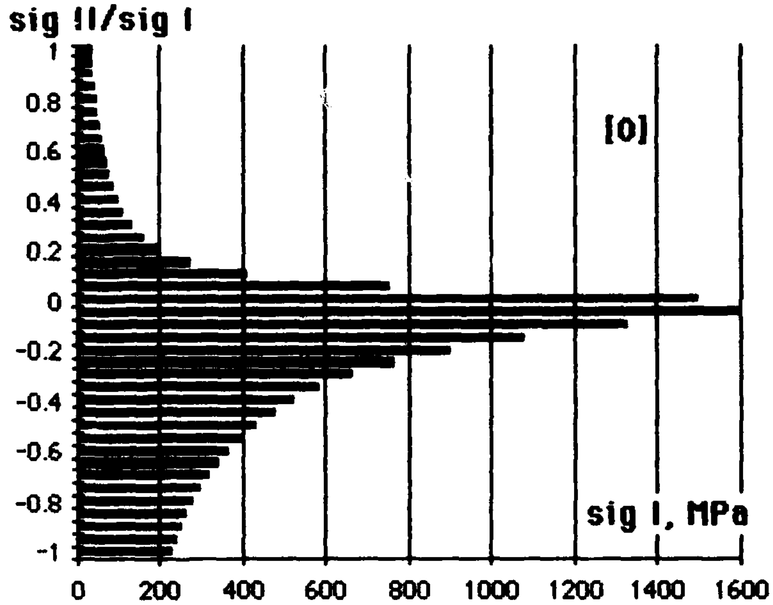


FIGURE 12.5 STRENGTH OF [0] T300/5208 VERSUS STRESS RATIOS.

There is a factor of 4 difference between the horizontal scale of this figure for the 0-degree CFRP and that for the aluminum. The 40 percent lighter in density of the CFRP than aluminum will increase the specific strength of CFRP over aluminum by a factor of 10.

This figure is limited to the following values of stresses:

$$\sigma_I > \sigma_{II}, \text{ and } \sigma_I > 0. \tag{12.5}$$

We did not show the compressive stress in the longitudinal direction because for this composite the transverse compressive strength is much higher than the tensile strength. The advantage of CFRP over aluminum is even greater for the case of $\sigma_I < 0$.

The highly directional nature of this CFRP and composites in general represents an opportunity. Our design concept is simply orient the material in the principal axes of the applied stress. Preferred orientation for isotropic materials does not exist. If we rotate our composite laminates to the principal stress direction, they will be anisotropic in the new laminate axes. We will have shear coupling resulting in beneficial or detrimental strains. As mentioned earlier the highly directional material has one inherent weakness: its precipitous loss in strength when the actual stress deviates from that anticipated in design.

12.5 SAMPLE PROBLEMS -

Prob 1: Find strength ratios for aluminum and CFRP T300/5208 for an applied load of $\sigma_i = \{100, 100, 100\}$, MPa.

SOLUTION:

Given the following state of stress:

$$\sigma_i = \{100, 100, 100\} \text{ MPa} \tag{12.6}$$

First we must find the principal stresses and the phase angle using Equations 5.10 and 5.16:

$p = 100$, $q = 0$, $r = 100$, $R = 100$

therefore $\sigma_I = 200$, $\sigma_{II} = 0$, $\theta_0 = [\arctan(100/0)]/2 = 45$ degree.

Using the definition of strength ratio in Equation 10.2 and the von Mises failure criterion in Equation 12.2 for aluminum:

$$R^{alum} = 1 \tag{12.7}$$

The strength ratio for our CFRP is simply the ratio between the longitudinal strength X (=1500 MPa) and the applied stress:

$$R^{cfrp} = 1500/200 = 7.5 \tag{12.8}$$

$$\text{Phase angle } \theta_0 = +45 \text{ degree} \tag{12.9}$$

This is the required rigid body rotation from the laminate axis in order to take advantage of the directional strength of CFRP.

The angle of rotation is positive or forward swing in Figure 5.3.

If the applied stress has negative shea, , i.e.,

$$\sigma_i = \{100, 100, -100\} \text{ MPa} \tag{12.10}$$

Then $p = 100$, $q = 0$, $r = -100$, $\theta_0 = -45$ degree.

Phase angle θ_0 = -45 degree (12.11)

The angle of rotation is negative or swept backward.

In Figure 12.6 we plotted the state of stress in both aluminum and CFRP to scale. The 7.5:1 ratio in their relative strengths is shown by the different sizes of the two Mohr's circles.

For composite materials the superior longitudinal strength can be utilized by orienting the material axis. The required rigid body rotations or forward or backward swing are shown in the figure for various combined stresses. Those for the plus and minus 45 degree are shown in Equations 12.9 and 11.11, respectively, multiplied by the strength ratio of 7.5 in Equation 11.8 to reflect the strength.

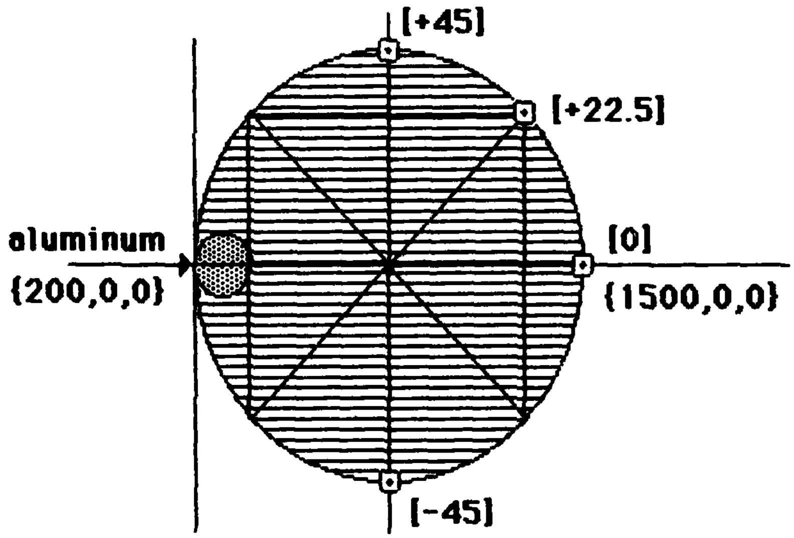


FIGURE 12.6 MOHR'S CIRCLE PLOTS OF UNIAXIAL STRENGTH OF ALUMINUM AND T300-5208, OR STRESS RATIO = 0. COMBINED STRESS CAPABILITY AND THE SWING ANGLES ARE ALSO SHOWN.

Prob 2: ***What are the resulting strains of our rotated composite? Is shear coupling beneficial or not?***

For the applied stress in Equation 12.6, we can easily compute the resulting strain from the compliance of aluminum, and T300/5208 in Table 6.4:

For aluminum, E = 69 GPa, ν = .3, G = 69/2.6 = 26.5 GPa

$$\epsilon_1 = \epsilon_2 = (1/69-.3/69)\text{x}100 = 0.001014$$
$$\epsilon_6 = 100/26.5 = .00377 \qquad (12.12)$$

For T300/5208 at θ = 45 degree,

$$\epsilon_1 = \epsilon_2 = (59.57-9.99-45.78)\text{x}100 = .00038$$
$$\epsilon_6 = (-45.78-45.78+105.71)\text{x}100 = .0014 \qquad (12.13)$$

Although the strain components are much smaller for the CFRP, it is better to compare the effective strain from a strain invariant. Among numerous invariants we may select for the present purpose:

$$\epsilon^{eff} = [\epsilon_1{}^2+\epsilon_2{}^2+\epsilon_6{}^2/2]^{1/2}$$

For aluminum: $\epsilon^{eff} = [2\text{x}.001041^2+.00377^2/2]^{1/2}$

$$= .003045 \qquad (12.14)$$

For CFRP: $\epsilon^{eff} = [2\text{x}.00038^2+.0014^2/2]^{1/2}$

$$= .001126 \qquad (12.15)$$

The ratio of $\epsilon^{al}/\epsilon^{cfrp} = 2.70$ (12.16)

CONCLUSION: the resulting strain is smaller and shear coupling beneficial.

If we make a mistake in swinging backward (-45 degree):

For CFRP at -45, $S_{16} = S_{26} = +45.78$:

$$\epsilon_1 = \epsilon_2 = (59.75-9.99+45.78)\text{x}100 = .00815$$
$$\epsilon_6 = (2\text{x}45.78+105.71)\text{x}100 = .0197 \qquad (12.17)$$

$$\epsilon^{eff} = [2\text{x}.00815^2+.0197^2/2]^{1/2} = .01808 \qquad (12.18)$$

This is higher than that in Equation 12.15 by a factor of 17.

CONCLUSION: Wrong swing angle is catastrophic!

Prob 3: Can we use the unidirectional CFRP for a load different from the uniaxial stress shown in Figure 12.6? How would a stress ratio of -0.2 perform relative to aluminum?

SOLUTION:

From Figure 12.5, we can find that:

$\sigma_2/\sigma_1 = -0.2$, the effective strength $\sigma_1 = 900$ MPa. (12.19)

The combined stress capability is {900,-180,0)

The combined stress capability of aluminum is (170,-34,0)

The relative strength is illustrated in Figure 12.7 which is comparable to the relative Mohr's circles in Figure 12.6.

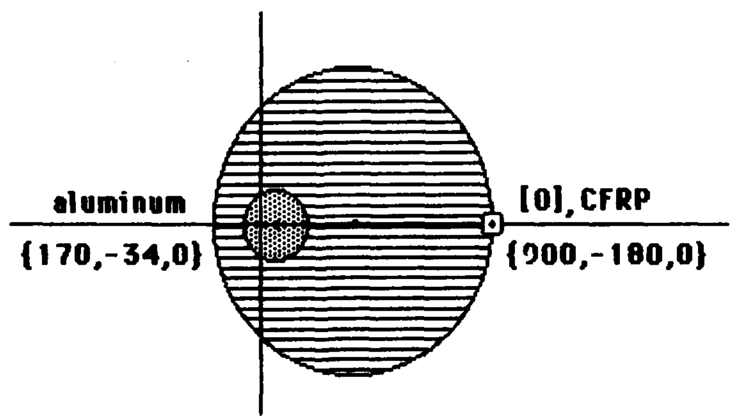


FIGURE 12.7 SAME SCALE AS FIG 12.6 EXCEPT STRESS RATIO = -0.2.

The advantage of CFRP over aluminum although smaller is still obvious.

CONCLUSION: Unidirectional composites are capable of carrying combined stresses effectively with proper laminate axes rotation;i.e., forward or backward swing.

Prob 4: What are the strength capability of CFRP laminates?

SOLUTION:

Highly directional properties can also be derived from simple laminates such as angle-ply and tridirectional laminates. Typical results for CFRP T300/5208 are shown below:

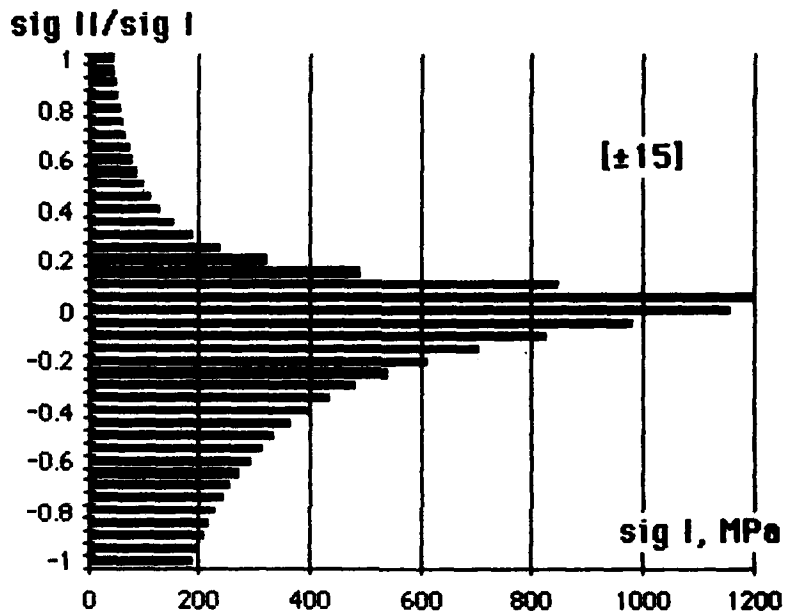


FIGURE 12.8 STRENGTH OF [±15] CFRP FOR VARIOUS STRESS RATIOS

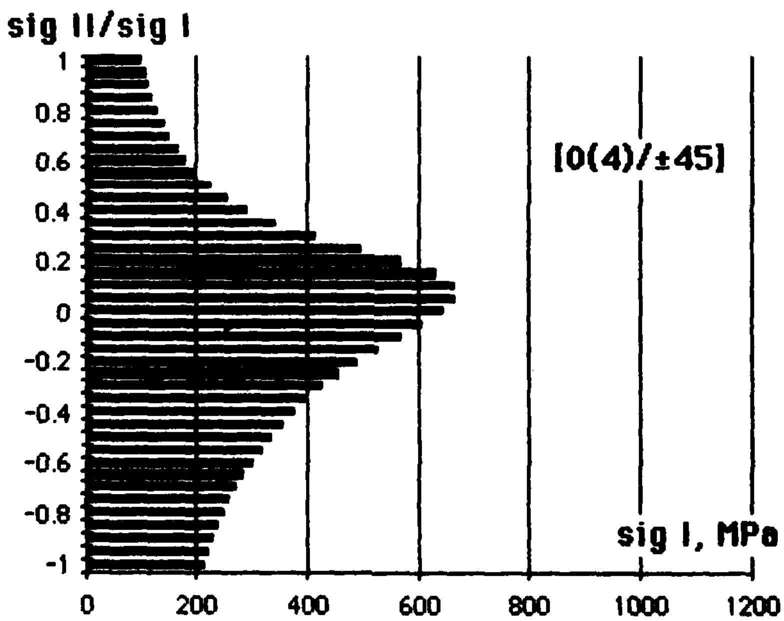


FIGURE 12.9 STRENGTH OF [0(4)/±45] FOR VARIOUS STRESS RATIOS

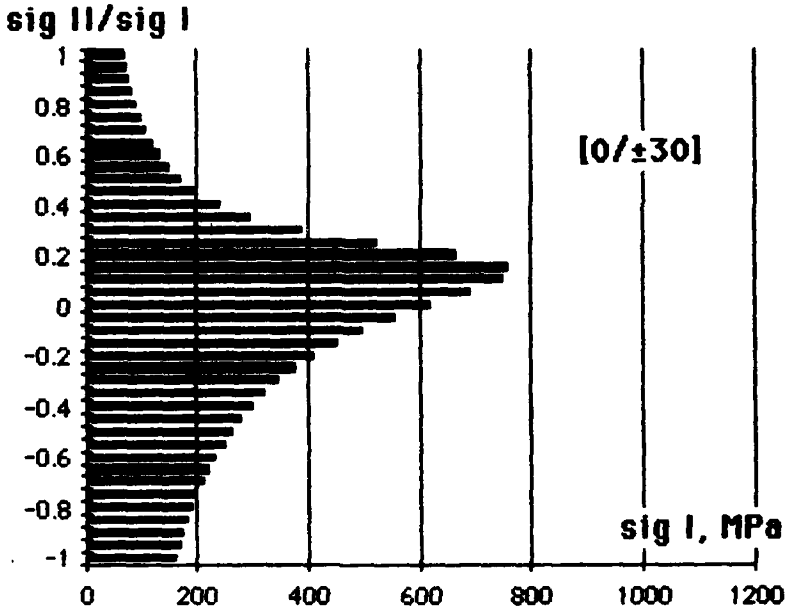


FIGURE 12.10 STRENGTH OF [0/±30] CFRP FOR VARIOUS STRESS RATIOS.

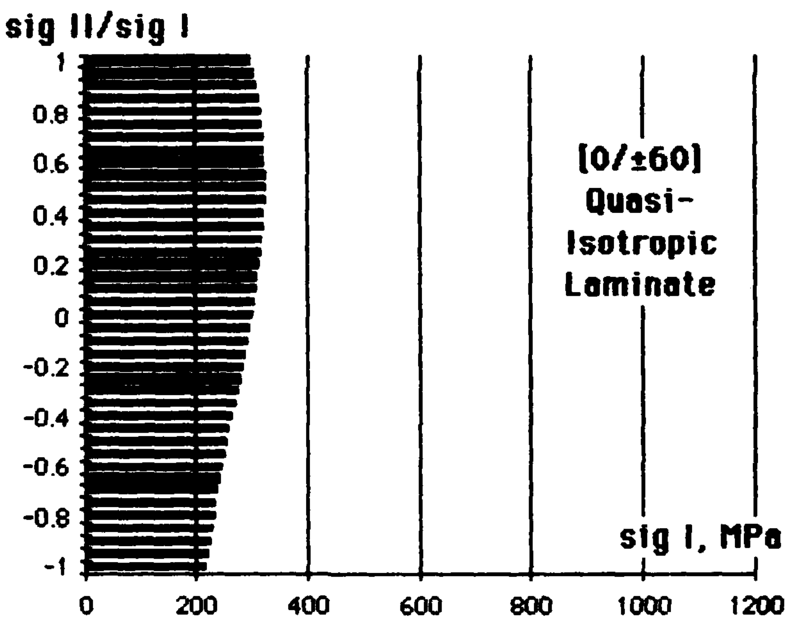


FIGURE 12.11 STRENGTH FOR CFRP QUASI-ISOTROPIC LAMINATES.

All the laminates with the exception of the quasi-isotropic laminate can be used for principal stress design. Only the highly directional ones show unusual benefit. But the less directional laminates are more suitable for multiple or wide-spectrum loads.

Section 13

DESIGN BY LAMINATE RANKING

13.1 LAMINATE RANKING – Another method of design can be based on the ranking a family or families of laminates. This is an alternative to the conventional optimization. The method in this section includes:

- Laminated plate theory of Section 7.
- Quadratic failure criterion of Section 10, with -0.5 for the normalized interaction term.
- Laminate strength base on the first-ply-failure.
- Simplified theory of thin wall construction in Section 9.
- Ambient temperature case only; with zero curing stresses.
- Single and multiple loads.
- CFRP T300/5208 only.
- A finite number (93) of sub-laminates divided in three families using 0, 90, 45 and -45 ply angles and having at least one ply in each of the selected angles:

  35 8-ply quadridirectional (all four angles) sub-laminates.

  40 6-ply tridirectional (any three out of four) sub-laminates.

  18 4-ply bidirectional (any two out of three) sub-laminates.

A new designation of 4211 means $[0_4/90_2/45/-45]$; 1023, $[0/45_2/-45_3]$.

13.2 SINGLE LOADS - In Section 12 laminates can be designed by the principal stress approach for single loads. We can use the laminate ranking to achieve similar results. The two approaches are complimentary. Both are based on the same theory. But we may arrive at different answers because of the different constraint imposed on the laminate families.

The principal stress approach is dependent on the particular basic laminates selected; i.e., [0], [±15], etc, and the rigid body rotation or the swing angle. For a given load we have to explore several laminates to arrive at an optimum construction. In general this approach is effective for the highly directional laminates. When the anticipated load is not well defined, a less directional composite may be preferred to reduce the sudden drop in strength. The laminates in Sub-section 12.6 are intended to show the range of options for the principal stress design. They also show the superior strength and stiffness capability of composites over isotropic materials.

The ranking approach is also limited by the family of laminates selected to be ranked. The size of the family is a function of speed and capacity of the computer. But multiple loads can be treated by this approach which cannot be easily done by the principal stress approach. In addition the ranking approach gives the options of having different laminates for the same combination of loads.

We will show the results of our ranking method for a number of simple and representative loads. Most results are difficult to predict. We therefore think that calculation is better than intuition or guessing.

Our strength design is based on the first-ply-failure and is conservative. The ranking is applied to both the number of plies required to carry the load and the ratio of plies of the laminate over the quasi-isotropic laminate. The number of plies are not always multiples of the sub-laminates. Adjustments must be made for the final laminate. This ratio over isotropy shows the advantage of anisotropy and can also be used to compare the strength advantage over isotropic materials such as aluminum and steel directly.

An explanation of the result and its display of the ranked laminates is shown in Figure 13.1.

APPLIED LOAD, AND NUMBER OF PLIES REQUIRED FOR THE LOAD

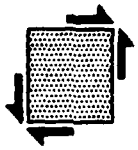
WEIGHT ADVANTAGE OVER QUASI-ISOTROPIC LAMINATE

LAMINATE DESIGNATION: PLY ANGLES AND SYMMETRY

NUMBER OF PLIES IN 0, 90, 45, AND -45

ID AND SORTING INDICES

APPLIED LOAD



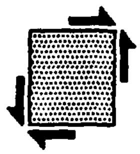
-3			N1	0
-2			N2	0
-1	# plies	laminate	N6	10
0			/iso	# req'd
91	0031	2 Aniso	1.82	108
92	0022	2 Ortho	1.72	114
63	1032	3 Aniso	1.60	122

FIGURE 13.1 EXPLANATION OF THE LAMINATE RANKING DISPLAY.

In Figure 13.2, we show the result of laminate ranking for pure shear. The best laminate is:

$$0031 \text{ or } [45_3/-45] \qquad (13.1)$$

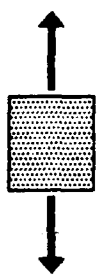
For an applied shear of 10MN/m, 108 plies are needed, which means 81 plies at 45 degrees and 27 plies at -45 degrees. The best result for the laminate is to use 27 repeating units of the sub-laminate.



-3			N1	0
-2			N2	0
-1	# plies	laminate	N6	10
0			/iso	# req'd
91	0031	2 Aniso	1.82	108
92	0022	2 Ortho	1.72	114
63	1032	3 Aniso	1.60	122
73	0132	3 Aniso	1.60	122
32	1142	4 Aniso	1.48	133
62	1041	3 Aniso	1.44	136
72	0141	3 Aniso	1.44	136
33	1133	4 Ortho	1.37	143
31	1151	4 Aniso	1.36	144
64	1023	3 Aniso	1.28	153

FIGURE 13.2 TOP 10 CFRP LAMINATES FOR PURE SHEAR [0,0,10]

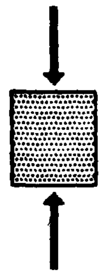
The case of uniaxial tension is shown in Figure 13.3. Since our laminate family does not include on- and off-axis undirectional composites, the result in this figure does not show what would be the highest ranking composite for uniaxial tension, the [0]. The best laminate among our family of 93 is 4011 which has 67 percent 0-degree plies. This laminate is better that the next 3 laminates although the latter all have 75 percent 0-degree plies. Had we use netting analysis, the higher percent 0-degree plies would give us higher laminate strength. The difference between our approach and the netting analysis is that we use laminated plate theory and the first-ply-failure criterion for laminate failure; whereas netting analysis is not an analysis and should not be used.



-3			N1	10.0
-2			N2	0.0
-1	# plies	laminate	N6	0.0
0			/iso	# req'd
56	4011	3 Ortho	2.29	62
79	3010	2 Aniso	2.25	63
82	3001	2 Aniso	2.25	63
76	3100	2 Ortho	1.91	74
57	3021	3 Aniso	1.78	79
58	3012	3 Aniso	1.78	79
1	5111	4 Ortho	1.75	81
36	4110	3 Aniso	1.73	82
46	4101	3 Aniso	1.73	82
80	2020	2 Aniso	1.52	93

FIGURE 13.3 TOP 10 CFRP LAMINATES FOR UNIAXIAL TENSION (10,0,0).

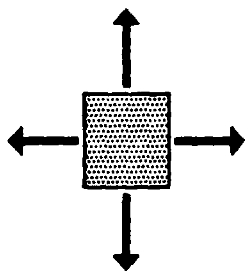
In Figure 13.4, we show the ranking of laminates for uniaxial compression. The best laminate now is 3100 having 75 percent 0-degree plies. This is better than the next two 4110 or 4101 having 67 percent 0-degree plies. While this may be consistent with netting analysis, the apparent consistency breaks down as the percentage of 0-degree plies increase. We can show that cross-ply laminate with small amount of 90-degree plies have higher strength than all 0-degree plies. The message here is the same about designing laminates: do not guess or use intuition.



-3			N1	-10
-2			N2	0
-1	# plies	laminate	N6	0
0			/iso	# req'd
76	3100	2 Ortho	2.69	26
36	4110	3 Aniso	2.04	34
46	4101	3 Aniso	2.04	34
77	2200	2 Ortho	1.92	36
1	5111	4 Ortho	1.78	39
2	4211	4 Ortho	1.68	41
79	3010	2 Aniso	1.68	41
82	3001	2 Aniso	1.68	41
37	3210	3 Aniso	1.67	41
47	3201	3 Aniso	1.67	41

FIGURE 13.4 TOP 10 CFRP LAMINATES FOR UNIAXIAL COMPRESSION [-10,0,0].

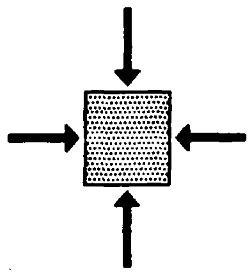
For two-dimensional hydrostatic pressure, the quasi-isotropic is the best laminate, followed by cross-ply and ±45-degree angle-ply. This is shown in Figure 13.5.



-3			N1	10
-2			N2	10
-1	# plies	laminate	N6	0
0			/iso	# req'd
15	2222	4 Q-ISO	1.00	128
77	2200	2 Ortho	0.97	132
92	0022	2 Ortho	0.97	132
5	3311	4 Ortho	0.96	134
33	1133	4 Ortho	0.96	134
6	3221	4 Aniso	0.82	157
7	3212	4 Aniso	0.82	157
12	2321	4 Aniso	0.82	157
13	2312	4 Aniso	0.82	157
18	2132	4 Aniso	0.81	158

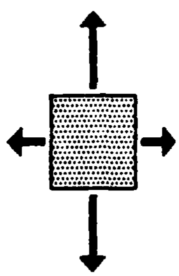
FIGURE 13.5 TOP 10 CFRP LAMINATES FOR HYDROSTATIC TENSION [10,10,0].

The ranking for hydrostatic compression in Figure 13.6 is the same as that for hydrostatic tension in Figure 13.5 except the number of plies are different. This would be expected because transverse compressive strength is four times higher than transverse tensile. The laminate strength turns out to be six times higher.



-3			N1	-10
-2			N2	-10
-1	# plies	laminate	N6	0
0			/iso	# req'd
15	2222	4 Q-ISO	1.00	20
77	2200	2 Ortho	0.97	20
92	22	2 Ortho	0.97	20
5	3311	4 Ortho	0.96	21
33	1133	4 Ortho	0.94	21
6	3221	4 Aniso	0.71	28
7	3212	4 Aniso	0.71	28
12	2321	4 Aniso	0.71	28
13	2312	4 Aniso	0.71	28
32	1142	4 Aniso	0.70	28

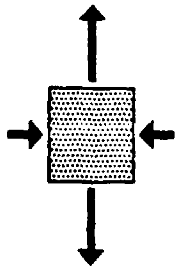
FIGURE 13.6 TOP 10 CFRP LAMINATES FOR HYDROSTATIC COMPRESSION (-10.-10.0).



-3			N1	10
-2			N2	5
-1	# plies	laminate	N6	0
0			/iso	# req'd
60	2022	3 Ortho	1.33	96
9	3122	4 Ortho	1.26	101
2	4211	4 Ortho	1.24	103
76	3100	2 Ortho	1.15	111
3	4121	4 Aniso	1.12	113
4	4112	4 Aniso	1.12	113
63	1032	3 Aniso	1.11	115
64	1023	3 Aniso	1.11	115
77	2200	2 Ortho	1.10	116
1	5111	4 Ortho	1.09	117

FIGURE 13.7 TOP 10 CFRP LAMINATES FOR INTERNAL PRESSURE OF A CYLINDRICAL VESSEL (10.5.0).

Two bi-axial loadings are shown in Figures 13.7 and 13.8. In the former, a 2:1 biaxial tension is presented. This corresponds to the stress induced in a cylindrical pressure vessel by internal pressure. In the latter, biaxial tension and compression with ratio of 2:-1 is presented:



-3			N1	10
-2			N2	-5
-1	# plies	laminate	N6	0
0			/iso	# req'd
76	3100	2 Ortho	1.99	83
36	4110	3 Aniso	1.69	97
46	4101	3 Aniso	1.69	97
1	5111	4 Ortho	1.54	107
77	2200	2 Ortho	1.54	107
37	3210	3 Aniso	1.53	108
47	3201	3 Aniso	1.53	108
2	4211	4 Ortho	1.50	110
38	3120	3 Aniso	1.42	116
48	3102	3 Aniso	1.42	116

FIGURE 13.8 TOP 10 CFRP LAMINATES FOR {10,-5,0}.

Finally, single load design by laminate ranking can be combined with that by principal stress. The laminate selected from ranking can be rotated or swept in the principal stress design. The two methods are really complimentary.

13.3 MULTIPLE LOADS – Multiple loads can be viewed several ways. Traditionally it deals with different loads at a point in a structure. Each load corresponds to the conditions that the structure is likely to be exposed to during its manufacturing, installation, operation, and maintenance. Multiple loads can also be viewed as the variation of loads at different locations within a structure where the ply number and angles remain constant. Multiple loads design can also provide a rational basis to select smooth transition of ply drop-off (change in number of plies) from point to point within a structure.

The explanation of the displayed result of our laminate ranking is shown in Figure 13.9. The display is very similar to that for the single load except a column is needed for each additional load. In addition, the maximum plies required for each laminate of the family is the controlling load. The laminates for the controlling load are ranked in ascending order. The first laminate is the lightest one from strength consideration.

MULTIPLE LOADS, AND NUMBER OF PLIES REQUIRED

WEIGHT ADVANTAGE OVER QUASI-ISOTROPIC

MAXIMUM PLIES REQUIRED

PLY NUMBER AND CODE

ID INDEX

1 2 3

-3			-3	N1	10	0	10
-2			-2	N2	0	5	5
-1	# plies	laminate	-1	N6	0	0	0
0			0	/iso	# req'd	# req'd	# req'd
2	4211	4 Ortho	103	1.37	96	82	103
76	3100	2 Ortho	111	1.28	74	97	111

FIGURE 13.9 EXPLANATION OF THE LAMINATE RANKING FOR MULTIPLE LOADS.

In Figure 13.10 we show a multiple loads of (10,0,0), (0,5,0) and (10,5,0):

-3			-3	N1	10	0	10
-2			-2	N2	0	5	5
-1	# plies	laminate	-1	N6	0	0	0
0			0	/iso	# req'd	# req'd	# req'd
2	4211	4 Ortho	103	1.37	96	82	103
76	3100	2 Ortho	111	1.28	74	97	111
9	3122	4 Ortho	111	1.27	111	101	101
3	4121	4 Aniso	113	1.25	95	113	113
4	4112	4 Aniso	113	1.25	95	113	113
77	2200	2 Ortho	116	1.22	107	54	116
6	3221	4 Aniso	122	1.16	122	77	121
7	3212	4 Aniso	122	1.16	122	77	121
5	3311	4 Ortho	123	1.15	122	61	123
1	5111	4 Ortho	123	1.15	81	123	117

FIGURE 13.10 TOP 10 CFRP LAMINATES FOR MULTIPLE LOADS OF (10,0,0), (0,5,0), AND (10,5,0).

In the figure we list the resulting laminate ranking. The controlling load for each laminate is highlighted. It is impossible to guess the ranking and the controlling load of laminates.

This example of multiple load condition further illustrates that all loads must be considered simultaneously. We do not know ahead of time what is the controlling load and how multiple loads interact. Had we designed our laminate using netting analysis or the carpet plot, one implicit assumption would be that loads do not interact. We can size laminates by one load or stress component at a time.

For the axial load of 10 MN/m in the 1-direction, the best laminate would be, from Figure 13.3, laminate 4011 with 62 plies. For the second load only, the top 10 laminates are entirely different from those for the first load. Finally, if both first and second loads occur simultaneously, the top 10 laminates for this load #3 are again different. These three loads when operate individually can have the following top 10 laminates:

LOAD 1 ONLY		LOAD 2 ONLY		LOAD 3 ONLY	
N1	10		0		10
N2	0		5		5
N6	0		0		0
# plies	# req'd	# plies	# req'd	# plies	# req'd
4011	62	0411	31	2022	96
3010	63	0310	32	3122	101
3001	63	0301	32	4211	103
3100	74	1300	37	3100	111
3021	79	0321	39	4121	113
3012	79	0312	39	4112	113
5111	81	1511	40	1032	115
4110	82	1410	41	1023	115
4101	82	1401	41	2200	116
2020	93	0220	47	5111	117

But these laminates are again different if all 3 loads are present as multiple loads. The best laminates for this combination loads are shown in Figure 13.10. Again our message is very simple: calculate and avoid needless guessing. Simple superposition implied by netting analysis and carpet plots can lead to unreliable results.

13.4 BENDING OF THIN WALL CONSTRUCTIONS – As mentioned earlier, multiple load design can also be applied to a variety of design problems, which include the design of unsymmetric thin wall construction. The theory of this special construction has been convered in Section 9. We will now apply the laminate ranking method to this unsymmetric construction. The relation between the in-plane and flexural loads applied to the entire construction and the in-plane loads applied to the face sheets is shown in Equation 9.41 and repeated here:

$$\{N+\} = \{N/2+M/h\},\ \{N-\} = \{N/2-M/h\} \tag{13.2}$$

If we only have applied moments,

$$\{N+\} = -\{N-\} = \{M\}/h \tag{13.3}$$

Because of this simple relation between N+ and N−, it is easy to apply the laminate ranking for single or multiple moments by checking the strength ratios of both the top and bottom faces. If the top face results in a strength ratio R, the bottom face is simply R' mentioned as the conjugate root after Equation 10.13.

Such simple strength calculation is not applicable if we have simultaneously applied moments and in-plane loads. Then the in-plane loads applied to the top face are unrelated to those applied to the bottom face, not just a sign change as in Equation 13.3. But the ranking approach reamins valid. The strength ratio or the number of plies required must be calculated seperately for the top and the bottom faces.

In Figure 13.11, we show the unsymmetric thin wall construction where the thickness of core c must be nearly equal to, that is within 10 percent of, the total thickness of the construction h. The superscript + and − refer to the top and the bottom face sheets. The sub-laminates (SL+ and SL−) and repeating indices (r+ and r−) are shown. A sandwich core or a substructure of spars and ribs that can support the face sheets is assumed. If the construction is a "thick wall" variety, simple relations like those in Equations 13.2 and 13.3 do not exist. A full laminated plate theory in place of the simplified one will be needed.

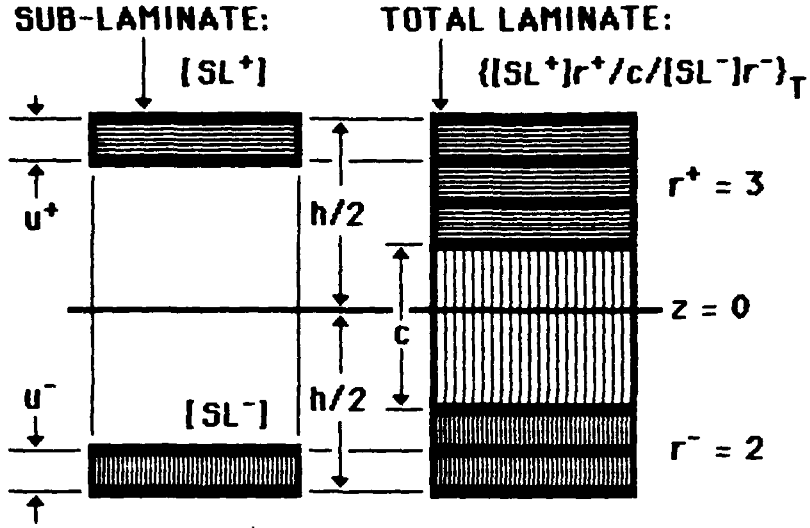
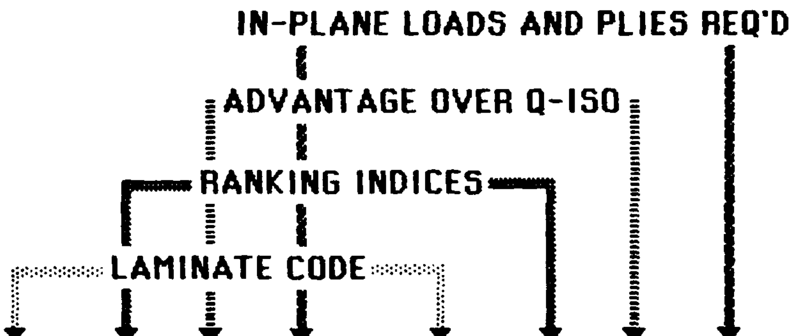


FIGURE 13.11 AN UNSYMMETRIC THIN WALL CONSTRUCTION.

The ranking process is the same as that for the in-plane loading. In Figure 13.12, the terminology of the lamnate ranking for bending is shown. Basically, the top and bottom face sheets are ranked seperately because the loads have opposite signs, as indicated in Equation 13.3. We only show one bending moment applied resolved into in-plane loads applied to the top and bottom faces on the left and right side in Figure 13.12, respectively.



BENDING	-3	N1+	10
ONLY	-2	N2+	0
# plies	-1	N6+	-2
	0	/iso	# req'
3001	61	2.42	61
2002	75	1.96	75

BENDING	-3	N1-	-10
ONLY	-2	N2-	0
# plies	-1	N6-	2
	0	/iso	# req'
2202	57	1.33	57
3201	62	1.23	62

FIGURE 13.12 EXPLANATION OF THE DISPLAY OF SINGLE MOMENT CASE. THE TOP FACE IS ON THE LEFT, AND THE BOTTOM, ON THE RIGHT.

The case of a single applied moment is displayed in Figure 13.13. The top and bottom faces are ranked individually. The lightest structure would be the lightest top and bottom faces. But other constraints may dictate the use of face laminate other than the lightest. But for the particular case, the weight advantage of the lightest is 61+57 = 118 plies. The next lightest combinations would be 61+62 = 123 plies, etc. Note that the applied moment results in a wider variation in the ply number in the top face than that in the bottom face. A wider choice of laminate is available for the bottom face without a sever weight penalty; i.e., the tenth ranking bottom laminate is only 8 plies more than the top ranked (65 versus 57).

BENDING	-3	N1+	10
ONLY	-2	N2+	0
# plies	-1	N6+	-2
	0	/iso	# req'
3001	61	2.42	61
2002	75	1.96	75
4011	90	1.63	90
3012	94	1.57	94
3102	95	1.54	95
4101	96	1.52	96
4112	101	1.46	101
3021	102	1.44	102
5111	104	1.42	104
3201	109	1.35	109

BENDING	-3	N1-	-10
ONLY	-2	N2-	0
# plies	-1	N6-	2
	0	/iso	# req'
2202	57	1.33	57
3201	62	1.23	62
2301	62	1.23	62
4121	63	1.21	63
3102	63	1.20	63
4211	64	1.20	64
3010	64	1.19	64
5111	64	1.19	64
3212	64	1.19	64
4112	65	1.17	65

FIGURE 13.13 TOP 10 CFRP LAMINATES UNDER A SINGLE APPLIED MOMENT. THE TOP FACE IS ON THE LEFT AND BOTTOM FACE ON THE RIGHT.

If the applied moment is fully reversible, the required number of plies for both faces will have to be identical. The result of the ranking is shown in Figure 13.14. We see that the positive moment, the first load, is the controlling one for all top 10 laminates. But the top 2 laminates for the top face and the top 7 for the bottom face in Figure 13.13 are not among the top 10 in Figure 13.14. Apparently, complete stress reversals demand different laminates to sustain the loads.

BENDING	-3	N1+	10	-10
ONLY	-2	N2+	0	0
# plies	-1	N6+	-2	2
	0	/iso	# req'	# req'
4011	90	1.63	90	73
3012	94	1.57	94	93
3102	95	1.54	95	63
4101	96	1.52	96	73
4112	101	1.46	101	65
3021	102	1.44	102	73
5111	104	1.42	104	64
3201	109	1.35	109	62
4121	114	1.29	114	63
4211	115	1.27	115	64

FIGURE 13.14 TOP 10 CFRP LAMINATES FOR THE SAME MOMENT AS THAT IN FIGURE 13.14 EXCEPT IT IS COMPLETELY REVERSIBLE.

In Figure 13.15 we show the case of designing two moments for our thin wall construction. Again we have only applied moments with no in-plane loads. The laminate ranking is applied seperately. It is apparent that guessing will be useless. The effects of the moments on the top and bottom faces are quite different. The first moment controls most of the top 10 laminates in the bottom face; and the second moment, the top face. The variation in the required ply number in the bottom face is again much less than that in the top face; i.e., 58-51 = 7 versus 85-59 = 26.

If the moments are reversible, totally or partially, the resulting in-plane loads applied to the top and bottom faces are treated as additional loads, in an identical manner as the case shown in Figure 13.13.

It may be more economic to use the same laminate for the top and bottom faces. For this multiple load case, 4110 laminate can serve both faces with 59+53 = 112 plies or 5111 laminate with 66+51 = 117 plies.

BENDING	-3	N1+	-4	7	BENDING	-3	N1-	4	-7
ONLY	-2	N2+	2	0	ONLY	-2	N2-	-2	0
# plies	-1	N6+	1	1	# plies	-1	N6-	-1	-1
	0	/iso	# req'	# req'		0	/iso	# req'	# req'
4110	59	1.70	54	59	5111	51	1.35	51	37
3120	65	1.55	49	65	4110	53	1.31	53	37
5111	66	1.54	66	65	4211	53	1.30	53	38
4121	68	1.49	59	68	3201	53	1.29	53	49
3210	71	1.43	49	71	4121	54	1.28	54	41
4211	74	1.35	51	74	4112	54	1.27	54	40
4112	75	1.35	68	75	3102	55	1.25	55	45
3122	80	1.26	64	80	4101	56	1.23	56	43
3131	83	1.22	59	83	4011	56	1.23	56	46
3221	85	1.19	49	85	2202	58	1.19	55	58

FIGURE 13.15 TOP 10 CFRP LAMINATES UNDER MULTIPLE MOMENTS. THE TOP FACE IS ON THE LEFT, AND BOTTOM FACE, THE RIGHT.

Section 14

GLOSSARY

ADHEREND - Plate adhesively bonded to another plate.

ADHESIVE - Substance capable of holding two surfaces together.

ADVANCED COMPOSITES - Composite materials with structural properties comparable to or better than aluminum; e.g., boron, graphite and Kevlar composites.

AGING - Loss of properties through time-exposure of elevated temperature, ultra-violet radiation, moisture or other hostile environments.

ALLOWABLES - Property values used for design with a 95 percent confidence interval: the "A" allowable is the minimum value for 99 percent of the population; and the "B" allowable, 90 percent.

AMBIENT CONDITIONS - Prevailing environmental conditions such as the surrounding temperature, pressure, and relative humidity.

ANISOTROPY - Material properties that vary with the orientation or direction of the reference coordinates. Composite materials, like wood, are anisotropic.

ASPECT RATIO - Length-to-width ratio of a rectangular plate, the length-to-diameter ratio of a discontinuous fiber, or the length-to-bundle diameter ratio of a bundle of parallel fibers.

ANGLE-PLY LAMINATE - Possessing equal plies with positive and negative angles. This bi-directional laminate is simple because it is orthotropic, not anisotropic. A [±45] is a very common angle-ply laminate. A cross-ply laminate is another simple laminate.

ANTISYMMETRY - Special symmetry with sign change between off-diagonal components; e.g., an unsymmetric angle-ply laminate.

AUTOCLAVE - Pressure vessel that can maintain temperature and pressure of a desired air or gas for the curing of organic-matrix composite materials.

AXISYMMETRY - Symmetry about an axis; i.e., isotropic in the plane normal to the axis, and this material is called transversely isotropic.

BFRA - Boron fiber reinforced aluminum.

BFRP - Boron fiber reinforced plastic.

B-STAGE - Intermediate or precured stage of a thermosetting resin. This stage provides easier handling and processing.

BALANCED LAMINATE - Where plies with positive angles are balanced by equal plies with negative angles. While angle-ply laminates have only one pair of matched angles, balanced laminates can have many pairs, plus 0 and 90 degrees. A balanced laminate is orthotropic in in-plane behavior, but anisotropic in flexural behavior.

BENDING MOMENT - Stress couple that changes curvature of a beam or plate.

BLEEDER CLOTH - Nonstructural, fiber glass cloth placed adjacent to the composite material part to absorb excess resin during cure, and is removed from the part after cure.

BONDED JOINT - Location where two adherends are bonded together with a layer of adhesive in between. A lap joint positions the adherends with an overlap; a scarf joint, with matched tapered sections; a stepped joint, through steps.

BORON FILAMENT - Made by vapor deposition of B_4C onto a tungsten substrate. Two filament diameters, 0.1 and 0.2 mm, are made. Two outstanding features of boron are: the high longitudinal compressive strength, and the relative ease of fabrication of boron/aluminum composite material.

BOUNDARY CONDITIONS - Load and environmental conditions that exist at the boundaries. Conditions must be specified to perform stress analysis.

BREATHER - Porous material, such as a fabric or mat, placed inside the vacuum bag to facilitate removal of air, moisture, and volatiles during cure.

BUCKLING - Unstable lateral displacement of a structural part such as a panel caused by excessive compression and shear. Microbuckling of fibers in a composite material can also occur under axial compression.

BUNDLE STRENGTH - Strength obtained from a test of parallel filaments, with or without organic matrix. The bundle test is often used in place of the tedious monofilament test.

CAUL PLATE - Smooth, metallic plate that is placed in contact with the laminate to insure uniform pressure and temperature during cure, and smooth finish after cure.

CFRP - Carbon or graphite fiber reinforced plastic. Letter G in GFRP is for glass, not graphite.

COCURE - Simultaneous curing and bonding of a composite laminate to another material or parts like honeycomb core, and stiffeners. Cocuring can reduce fabrication cost.

COLLIMATED - Rendered parallel, such as fibers in unidirectional tapes.

COMPLIANCE - Measurement of softness as opposed to stiffness of a material. It is a reciprocal of the Young's modulus, or an inverse of the stiffness matrix.

COMPOSITE OR COMPOSITE MATERIAL - Material that is made of two or more constituent materials. Although most natural and manmade materials are composites, our current interest covers only fiber-reinforced materials; e.g., uni and multidirectional filamentary composites in woven and nonwoven forms.

CONSTITUENT MATERIALS - Individual materials that make up the composite material; e.g., graphite and epoxy are the constituent materials of a graphite/epoxy composite material.

CONTRACTED NOTATION - A short-hand notation for stress, strain and material constants such as elastic moduli and strength parameters.

CORE - Inner, light weight material used in a sandwich panel. Metallic or composite facing materials are bonded to the core to form a sandwich panel.

COUNT - Number of warp and fill yarns per unit length; e.g., a fabric count of 24x26 in the English unit means 24 yarns per inch in the warp, and 26 in the fill.

COUPLING - Linking a side effect to a principal effect. Poisson coupling links the lateral contraction to an axial extension. For composite materials, an anisotropic laminate couples the shear to normal components; an unsymmetric laminate couples curvature with extension. These couplings are unique with composites and provide opportunity to perform extraordinary functions.

CRAZING - Formation of fine cracks in an organic matrix material. Cracks in composite laminates are also referred to as crazing even though the cracks could be in the matrix material as well as at the interface between the constituent materials.

CROSS-PLY LAMINATE - Special laminate that contains only 0 and 90 degree plies. This bi-directional laminate is orthotropic, and has nearly zero Poisson's ratio. The other simple bi-directional laminate is the angle-ply, which possesses one pair of balanced off-axis plies.

CURE - To change the properties of a thermosetting resin irreversibly by chemical reaction. Cure may be accomplished by addition of curing agents, with or without catalyst, and with or without heat.

CURVATURE - Geometric measure of the bending and twisting of a plate.

DAM - Ridge circumventing a mold to prevent resin runout during cure.

DEBOND - Area of separation within or between plies in a laminate, or between a bonded joint caused by contaimination, improper adhesion during processing, or damaging interlaminar stresses.

DEBULK - To reduce laminate thickness by application of pressure. The compaction is achieved by removing trapped air, vapor, and volatiles between plies.

DEFORMATION - Changes in size and shape of a body resulting from externally applied stresses, temperature change, and moisture absorption. Deformation in size is measured by the normal strain components; that in shape, by shear component.

DEGRADATION - Loss of property due to aging, corrosion, and repeated or sustained stress (fatigue or stress rupture). Composite materials are insensitive to any form of degradation.

DELAMINATION - Debonding process primarily resulting from unfavorable interlaminar stresses. Edge delamination however can be effectively prevented by a wrap-around reinforcement.

DESIGN - To select optimum ply number and orientations for a given composite laminate subjected to one or more sets of applied stresses. This is the antithesis of stress analysis.

DIMENSIONAL STABILITY - Zero or nearly zero deformation due to change in temperature and moisture content. This unique property of graphite/epoxy composite material is utilized for antennas in space.

DISPLACEMENT - Measure of the movement of a point on the surface and in the interior of a body.

DOUBLER - Area builtup for local reinforcement or repair.

DRAPE - Ability of woven and nonwoven composite materials to conform to complex curvature.

DWELL TIME - Period of time during cure that a laminate is held at elevated temperature prior to application of pressure.

ELASTIC RELATION - Fully reversible, single-valued stress-strain relation. Loading and unloading follow the same path; there is no hysteresis, or residual strain. Although nonlinear relation is admissible, the relation for composite materials is essentially linear.

END - Individual warp yarn, thread, monofilament, or roving. For glass fibers, an end contains 206 filaments.

ENGINEERING CONSTANTS - Measured directly from uniaxial tensile and compressive, and pure shear tests applied to unidirectional as well as laminated composites. Typical constants are the effective Young's modulus, Poisson's ratio, and shear modulus. Each constant is accompanied by letter or numeric subscripts designating the direction associated with the property.

ENVIRONMENTAL CONDITIONS - Prevailing temperature and humidity.

EPOXY - Thermosetting resin made by polymerization of an epoxide.

EXPANSION COEFFICIENT - Measurement of swelling or expansion of composite materials due to temperature change and moisture absorption.

FABRIC - Planar, woven material constructed by interlacing yarns, fibers or filaments.

FAILURE CRITERION - empirical description of the failure of composite materials subjected to complex state of stresses or strains. The most commonly used are the maximum stress, the maximum strain, and the quadratic criteria.

FAILURE ENVELOPE - Ultimate limit in combined stress or strain state defined by a failure criterion.

FIBER - Single filament, rolled or formed in one direction, and used as the principal constituent of woven and nonwoven composite materials. Most common fibers are glass, boron, graphite and Kevlar.

FIBER CONTENT - Percent volume of fiber in a composite material. Most common composites in use today have fiber content between 45 and 70 percent. Percent weight or mass of fiber is also used.

FICK'S EQUATION - Diffusion equation for moisture migration. This is analogous to the Fourier's equation of heat conduction.

FILAMENT - Continuous fiber with exceptional high specific stiffness and strength, and is the principal constituent of advanced composites.

FILAMENT WINDING - Automated process of placing filament onto a mandrel in prescribed patterns. The resin impregnation can be before or during the winding, known as prepreg or wet winding, respectively. The mandrel can subsequently be removed after curing of the composite material. Filament winding is most advantageous in building pressure vessels, pipes, drive shafts, or any device that is axisymmetrical.

FILL - Yarn oriented at right angles to the warp in a woven fabric.

FINISH - Material and process used to treat fibers to improve the interfacial bond between fiber and matrix.

FIRST-PLY-FAILURE - First ply or ply group that fails in a multidirectional laminate. The load corresponding to this failure can be the design limit load.

FLASH - Excess material that is formed at the parting line of a mold or die, or that is extruded from a closed mold.

FOURIER'S EQUATION - Diffusion equation commonly associated with the heat conduction in a body. Fick's equation is a special case, applied to the moisture migration and accumulation.

GELCOAT - Quick-setting resin used in the molding process to provide an improved surface for the composite; it is the first resin applied to the mold after the mold-release agent.

GENERALIZED HOOKE'S LAW - The most general linear elastic stress-strain relation for an anisotropic material from which materials with various types of symmetries can be derived.

GFRP - Glass fiber reinforced plastic. Graphite fiber reinforce plastic is called CFRP.

GLASS FILAMENT - Drawn from molten glass through platnium bushings. Most widely used filament in composite materials. Glass filament can be made into unidirectional plies, woven fabric, mats, and short-fiber composites like the sheet molding compound.

GLASSY TEMPERATURE - When the stiffness and strength of an organic resin undergo drastic reduction. This is also known as the glass transition temperature and can be the maximum use temperature.

GRAPHITE FILAMENT - Made from poly-acrylic-nitrile (PAN) filament after high temperature exposure and mechanical stretching. Other materials and processes are available but not as common.

HETEROGENEITY - On the micro level, local variation of constituent materials. On the macro level, ply by ply variation of materials or orientations.

HOMOGENEITY - Material uniformity within a body. In the mechanics of composite materials, micro and macro homogeneity is achieved by smearing the actual heterogeneity.

HONEYCOMB CORE - Made from metallic and nonmetallic materials, and when bonded with face sheets form a sandwich panel. The core is assumed to have no stiffness in the plane of the sandwich panel, and infinite stiffness normal to the panel.

HORIZONTAL SHEAR STRENGTH - Estimated from a short-beam-shear test. This test is approximate because the stresses calculated from the simple beam theory is not exact.

HYBRID - Composite with more than two constituents; e.g., a graphite/glass/epoxy hybrid. An intralaminar hybrid has hybrid plies made from graphite and glass filaments. An interlaminar hybrid has laminates made from two or more different ply materials.

HYGROTHERMAL EFFECT - Change in properties due to moisture absorption and temperature change.

INDICIAL NOTATION - Mathematical description of matrices and tensors like stress, elastic moduli, et al. using subscript and superscript indices.

INTERACTION - Same as coupling. For example, longitudinal tensile strength is affected by the presence of transverse stress. Similar interaction exists between the longitudinal buckling stress and the transverse or shear stress. As a rule, the interaction effects for composite materials are greater than the conventional isotropic material. All anticipated stresses should be considered simultaneously.

INTERFACE - Boundary or transition zone between constituent materials, such as the fiber/matrix interface, or the boundary between plies of a laminate. Debond at the microscopic or fiber/matrix interface can lead to fiber breakage and matrix cracking. Debond at the macroscopic or interlamilar interface can lead to delamination.

INTERLAMINAR STRESSES - Three stress components associated with the thickness direction of a plate. The remaining three are the in-plane components of the plate. Interlaminar stresses are significant only if the thickness is greater than 10 percent of the length or width of the plate. These stresses can also be significant in areas of concentrated loads, and abrupt change in material and geometry. The effects of these stresses are not easy to assess because 3-dimensional stress analysis and failure criterion are not well understood.

INVARIANT - Constant values for all orientations of the coordinate axes. Components of stress, strain, stiffness and compliance all have linear and quadratic invariants. For composite materials they represent directionally independent properties, and the bounds of stiffness and strength of multidirectional laminates.

ISOTROPY - Property that is not directionally dependent. Stiffness and strength of aluminum, for example, are isotropic and remain the same for all orientations of the coordinate axes. Composite laminates can be made isotropic in its in-plane stiffness; e.g., any [π/n] laminate with n greater than 2, which is referred to as quasi-isotropic.

KEVLAR FIBER - DuPont company trade name for an aramid fiber.

KFRP - Kevlar fiber reinforced plastic.

LAMINA - Ply or layer of unidirectional composite or fabric.

LAMINATE - Plate consisting of layers of uni or multidirectional plies of one or more composite materials. A laminate is usually assumed to be thin and is under a state of plane stress.

LAMINATED PLATE THEORY - The most common method for the analysis and design of composite laminates. Each ply or ply group is treated as a quasi-homogeneous material. Linear strain across the thickness is assumed. This is also called the lamination theory.

LAYUP - Hand or machine-operated process of ply by ply laying of a multidirectional laminate.

LAYUP - Ply stacking sequence or ply orientations of a laminate.

LINEAR RELATION - Straight line relation between variables; i.e., each output variable is unique and linearly proportional to an input variable.

LOAD - Force, or generalized force such as in-plane or flexural stress.

LOADING PATH - Locus of increasing load in stress or strain space.

LONGITUDINAL MODULUS - Elastic constant along the fiber direction in a unidirectional composite; e.g., longitudinal Young's and shear moduli.

MACROMECHANICS - Structural behavior of composite laminates using the laminated plate theory. The fiber and matrix within each ply are smeared and no longer identifiable.

MANDREL - Male mold used for filament winding. For pressure vessels the mandrel is made of salt or plaster, supported by a collapsable framing.

MATRIX - Material that binds the filaments or fabric to form a composite material. The most common matrices for organic composites are polyester and epoxy; for metal-matrix composites, alunimum.

MATRIX - Mathematical entity, consisting of rows and columns of numbers. In two dimensions, stress and strain are 1x3 matrices; and stiffness and compliance, 3x3 matrices.

MATRIX INVERSION - Algebraic operation to obtain compliance matrix from stiffness matrix, or vice versa. It is analogous to obtaining the reciprocal of a number.

MAXIMUM STRESS or MAXIMUM STRAIN - Failure criterion based on the maximum or ultimate value of each component of stress or strain.

MECHANICAL LOAD - Mechanically applied load as distinguished from cure- or environment-induced load.

MICROMECHANICS - Calculation of the *effective ply properties* as functions of the fiber and matrix properties. Some numerical approaches also provide the stress and strain within each constituent and those at the interface.

MID-PLANE - Middle surface of a laminate; usually the z=0 plane.

MODULUS - Elastic constants such as the Young's modulus, shear modulus, or stiffness moduli in general.

MOHR'S CIRCLE - Graphical representation of the variation of the stress and strain components resulting from rotating coordinate axes. Analogous representations for material property such as the stiffness and compliance of composite laminates can also be made.

MOISTURE ABSORPTION - Property of epoxy changes due to moisture absorption; it is detrimental because the glassy temperature of the epoxy is suppressed, and beneficial because swelling counteracts curing stresses.

MOISTURE DISTRIBUTION - Transient moisture profile changes very slowly with time. For temperature, a steady-state distribution can be attained in a short time. But for moisture concentration, only the first few plies from the exposed surface can change significantly with time. For the interior plies, months if not years will elapse before change takes place. This nonuniform distribution must be considered in the assessment of the effect of moisture on the properties of composite materials.

MOLD - Cavity on which a composite part is placed, and from which it takes its shape after curing.

MOLD RELEASE AGENT - Lubricant applied to mold surfaces to facilitate release of the molded part.

MOMENT - Stress couple that causes a plate to bend or twist.

MULTIDIRECTIONAL - Having multiple ply orientations in a laminate.

NOTCHED STRENGTH - The effective strength of a plate with stress raisers like holes, notches and cracks.

OFF-AXIS - Not coincident with the symmetry axis; also called off-angle.

ON-AXIS - Coincident with the symmetry axis; also called on-angle.

OPTIMUM LAMINATE - Having the highest stiffness or strength per unit mass or cost.

ORTHOTROPY - Having three mutually perpendicular planes of symmetry. Unidirectional plies, fabric, cross-ply and angle-ply laminates are all orthotropic.

PARALLEL-AXIS THEOREM - Formula for elastic moduli relative to a

displace reference coordinate; analogous to that for moment of inertia.

PEEL PLY - Fabric material applied to a laminate to protect the clean, ready-to-use bonding surface and peeled off prior to curing.

PHENOLIC - Thermosetting resin for elevated temperature.

PICK - Individual fill yarn or roving in a fabric.

PLANE STRAIN - Two-dimensional simplification for stress analysis, applicable to the cross-section of long cylinders.

PLANE STRESS - Two-dimensional simplification for stress analysis, applicable to thin homogeneous and laminated plates.

PLATEN - Mounting plates of a press to which the mold assembly is fastened.

PLY GROUP - Group formed by contiguous plies with the same angle.

PLY STRAIN - Those components in a ply which, by the laminated plate theory, are the same as those of the laminate.

PLY STRESS - Those components in a ply, which vary from ply to ply depending on the materials and angles in the laminate.

POROSITY - Having voids; i.e., containing pockets of trapped air and gas after cure. Its measurement is the same as void content. It is commonly assumed that porosity is finely and uniformily distributed throughout the laminate.

POST CURE - Additional exposure to temperature after initial cure. The post cure temperature may be higher than the cure temperature.

PREFORM - Layup made on a mandrel or mockup that is subsequently transferred to the curing tool or mold.

PREPREG - Short for preimpregnated; a woven or unidirectional ply, or roving impregnated with resin, usually advanced to B-stage, and ready for layup or winding.

PRINCIPAL DIRECTION - Specific coordinate axes orientation when stress and strain components reach maximum and minimum for the normal components, and zero for the shear.

REPEATING INDEX - Used in laminate code to represent the number of repeating sub-laminates.

RESIDUAL STRESS - Resulting from cooldown after cure and moisture content. On the micromechanical level, stress is tensile in the resin and compressive in the fiber. On the macromecahnical level, it is tensile in the transverse direction to the unidirectional fibers, and compressive in the longitudinal direction, resulting in a lowered first-ply-failure load. Moisture absorption offsets this detrimental thermal effect in both micro and macro levels.

RESIN - Organic material with high molecular weight, insoluble in water, with no definite melting point and no tendency to crystallize.

RESIN CONTENT - Percent resin in a composite material; by weight for the materials scientist, and by volume for the mechanics.

RESIN-RICH AREA - Where local resin content is higher than the average of a laminate due to improper compaction or curing. It may be detrimental because every resin-rich area may mean a resin-starved area somewhere else.

RESIN-STARVED AREA - Where local resin content is lower than the average. It has a dry appearance where filaments or fabric is not completely wetted. This is probably more detrimental than the resin-rich area.

RIGID-BODY ROTATION - Rotation without change in shape; occurs in the relation between the off- and on-axis failure envelopes if rotation is carried out in the Mohr's circle or the q-r plane.

ROSIN - Specific natural resin, not to be confused with resin for composite materials.

ROVING - Loose assemblage of filaments. Roving can be impregnated for use in filament winding, in braiding, and in unidirectional tapes.

RULE-OF-MIXTURES - Linear volume fraction relation between the composite and the corresponding constituent properties.

SANDWICH PANEL - Consisting of two thin face sheets bonded to a thick, lightweight, honeycomb or foam core.

SCRIM - Reinforcing fabric woven into an open mesh construction, used in processing of tape and other B-staged material to facilitate handling.

SECANT MODULUS - Idealized Young's modulus derived from a secant drawn between the origin and any point on a nonlinear stress-strain curve. Tangent modulus is the other idealized Young's modulus derived from the tangent to the stress-strain curve.

SECONDARY BONDING - Joining by adhesive bonding parts which have already been cured. This is different from cocuring.

SHEAR COUPLING - Induced shear strain from normal stress. This coupling is unique with anisotropic materials.

SHEAR STRESS - Component that results in distortion; different from normal components that results in extension or contraction. For multidirectional laminates, in-plane shear is present in every ply. This shear and the in-plane normal components are all discontinuous when ply orientation changes; but the discontinuity should cause no concern. Transverse shears are shear stresses associated with the thickness direction, not in the plane of the laminate. They are difficult to calculate, and equally difficult to observe experimentally; but are important only when plate thickness is greater than 10 percent of the length or width.

SHEET MOLDING COMPOUND (SMC) - Short-fiber-reinforced composite.

SHELF LIFE - Length of time a material can be stored under specified environmental conditions without failure to meet specifications.

SHRINKAGE - Contraction of a molded part during and after cure. Parts can meet dimensional tolerance if the dimensional changes due to temperature of the part and the mold are properly considered.

SIZING - To select by design the ply number and angles of a laminate subjected to one or more sets of applied stresses. Sizing of isotropic materials is easy because there is one thickness required for each load. Sizing of composite laminates is fundamentally different because both ply number and angles must be considered. It is a nonlinear process; i.e., 10 percent ply addition does not mean 10 percent increase in strength.

SQUARE-SYMMETRY - Having equal stiffness or strength properties in two orthogonal axes; e.g., a fabric with square weave.

STACKING SEQUENCE - Ply ordering in a laminate. Stacking sequence does not affect the in-plane properties of a symmetric laminate. Only the ply number and angles are important. But stacking sequence becomes critical for the flexural properties, and the interlaminar stresses for any laminate, symmetric or not.

STIFFNESS - Ratio between the applied stress and the resulting strain. Young's modulus is the stiffness of a material subjected to uniaxial stress; shear modulus, to shear stress. For composite materials, stiffness and other properties are dependent on the orientation of the material. They must be further identified with a direction, usually designated by subscripts such as x, y, s, or 1, 2, 6.

STRAIN - Geometric measurement of deformation.

STRENGTH - Maximum stress that a material can sustain. Like the stiffness of a composite material, strength is highly dependent on the direction as well as the sign of the applied stress; e.g., axial tensile, transverse compressive, and others.

STRENGTH PARAMETER - Strength coefficient of a quadratic failure criterion in stress or strain space; tensors F and G, respectively.

STRENGTH RATIO OR STRENGTH/STRESS RATIO - Useful measure related to margin of safety. Failure occurs when the ratio is unity; safety is assured for example by a factor of 2 if the ratio is 2. The ratio is particularly easy to obtain if the quadratic failure criterion is used.

STRESS - Intensity of forces within a body. The normal components induces length or volume change; the shear component, shape change. The numerical value of each component changes as the reference coordinate system rotates. For every stress state there exists a principal direction, a unique direction when the normal components reach maximum and minimum, and the shear component vanishes.

STRESS CONCENTRATION - Increased ratio of a local stress over the average stress. On the micromechanical level, concentration occurs at the fiber/matrix interface. On the macromechanical level, concentration occurs at notches, ply termination points, joints, etc.

STRESS-STRAIN RELATION - Linear relation is usually assumed for calculating stress from strain, or from strain to stress. For multidirectional laminates, it can be generalized to include in-plane stress-strain, and flexural stress-strain relations. All anisotropic relations are simple extensions of the isotropic relation.

SUB-LAMINATE - Repeating multidirectional assemblage within a laminate.

SUCCESSIVE PLY FAILURE - Sequential failures of plies in a multidirectional laminate due to increasing loads.

SWELLING - Volumetric increasing due to temperature increase or moisture absorption.

SYMMETRIC LAMINATE - Possessing midplane symmetry. This is the most common construction because the curing stresses are also symmetric. The laminate does not twist when the temperature and moisture content change. An unsymmetric laminate on the other hand twists upon cooldown and untwists after moisture absorption.

SYMMETRIC MATRIX - Equal off-diagonal components in a matrix; i.e., components "12" is equal to "21"; or "ij" equal to "ji" for the entire range of values of the i and j subscripts. The stiffness and compliance matrices for all materials including composite materials are always symmetric. This symmetry is also known as the reciprocal relation.

SYMMETRY IN MATERIAL - Repeating material property; four common symmetries for composite materials are: orthotropy, transverse isotropy, square symmetry, and, ultimately, isotropy. For these cases the functional relations between stress and strain remain the same, only the independent material constants decrease from 9, 5, 3, to 2, respectively.

SYMMETRY IN PLY STACKING - Midplane symmetry in ply stacking or layup of a laminate; resulting in a symmetric laminate.

SYMMETRY IN TRANSFORMED PROPERTIES - Transformed stiffness and compliance components of an orthotropic material have even and odd symmetries. The diagonal components are even; i.e., components 11, 22 and 66 are always positive and symmetrical with respect to the 0 and 90 degree axes. The Poisson coupling component is positive for the stiffness matrix; negative, for the compliance. Remaining components 16 and 26 are odd; i.e., their value can be either positive or negative.

TACK - Stickiness of a prepreg; an important handling characteristic.

TANGENT MODULUS - Idealized Young's modulus derived from the tangent drawn at the origin or any point on a nonlinear stress-strain curve. Secant modulus is the other idealized Young's modulus between the origin and same point on the stress-strain curve.

TEMPLATE - Pattern used as a guide for cutting and laying plies.

THERMAL LOAD - One component of the hygrothermal load. The difference between the cure and use temperature gives rise to in-plane thermal load for symmetric laminates; and both in-plane and flexural thermal loads for unsymmetric laminates. The presence of the flexural load causes twisting of unsymmetric laminates after cure.

THERMOPLASTIC - Organic material that can reversibly change its stiffness by temperature change. One unique property of this material is its large strain capability. But processing requires high temperature and pressure, compared with thermosetting plastics.

THERMOSETTING PLASTIC - Organic material that can be converted to a solid body by cross-linking, accelerated by heat, catalyst, ultraviolet light, and others. This is the most popular matrix material for composite materials.

TOW - A loose, untwisted bundle of filaments. Graphite fibers are available in multiples of 3000-filament tows.

TRANSFORMATION - Variation of stiffness, strength, hygrothermal expansion, stress, strain and others due to the coordinate transformation or, simply, the rotation of the reference coordinate axes. Transformation follows strict mathematical equations. The study of composite materials relies heavily these transformation equations to correctly describe the directional dependency of the materials. Mohr's circles are geometric representation of the transformation equations. Associated with each transformation are several invariants which are useful design parameters.

TRANSVERSE CRACK - Matrix and interfacial failure caused by excessive tensile stress applied transversely to the fibers in a unidirectional ply of a laminate. This cracking is normally the source of the first-ply-failure.

TRANSVERSE ISOTROPY - Material symmetry that possesses an isotropic plane; e.g., a unidirectional composite.

UNIDIRECTIONAL COMPOSITE - Having parallel fibers in a composite.

UNSYMMETRIC LAMINATE - One without mid-plane symmetry.

VACUUM BAG - Flexible nylon, Mylar or elastomeric film that provides the outer cover of composite material cure assemblage, which can be sealed and evacuated to provide compaction pressure. The entire assemblage with the bag can be placed in an oven or autoclave for curing at desired temperature and additional pressure.

VOLUME FRACTION - Fraction of a constituent material based on its volume.

VOID - Air or gas trapped in a composite materials during cure.

VOID CONTENT - Volume percentage of voids, usually less than 1 percent. The experimental determination is however only indirect; i.e., calculated from the measured density of a cured composite and the "theoretical" density of the starting material. Such determination also implies that voids are uniformily distributed throughout the body.

WARP - Yarn oriented longitudinally in a fabric, and perpendicularly to the fill yarn.

WEIBULL PARAMETERS - Frequently used statistical measures for the static and fatigue strengths of composite materials. Shape parameter α is proportional to the coefficient of variation; scale parameter β, to the mean.

WET - Having absorbed moisture. Like having voids, a uniform moisture distribution is implied which is unlikely to attain in real life. Most absorbed moisture is located near the exposed surface. The interpretation of the effect of being wet must take into account the realistic, highly nonuniform moisture distribution.

YOUNG'S MODULUS - The slope of a stress-strain curve under uniaxial test.

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